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A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE

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CRITERIA FOR THE IRREDUCIBILITY OF FUNCTIONS IN A FINITE FIELD.

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BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, September 3, 1906.)

1. **Theorem.** *A necessary condition that*

$$(1)\qquad f(x) \equiv x^m + c_1 x^{m-1} + \cdots + c_m$$

shall be irreducible in the $GF[p^n]$, $p > 2$, is that its discriminant be a square or a not-square according as m is odd or even.*

If $f(x) = 0$ is irreducible its roots are $\lambda^{p^{ni}}$ $(i = 0, 1, \cdots, m-1)$. Its discriminant is therefore the square of P, where

$$P = \prod_{i<j}^{i,j=0,1,\ldots,m-1} (\lambda^{p^{ni}} - \lambda^{p^{nj}}) \equiv \prod f_{i,j}.$$

For $j < m-1$, we have $f_{i,j}^{p^n} = f_{i+1,j+1}$. But $f_{i,m-1}^{p^n} = -f_{0,i+1}$. Hence

$$P^{p^n} = (-1)^{m-1} P,$$

so that P equals a mark of the $GF[p^n]$, $p > 2$, if and only if m is odd.

Remark. The condition is also sufficient if $m = 2$.

2. **Lemma.** *The necessary and sufficient condition that a cubic shall have one and but one root in the $GF[p^n]$, $p > 2$, is that its discriminant be a not-square.*

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* As in the theory of algebraic equations, it is here convenient to designate as the discriminant the product of the squares of the differences of the roots. Most writers on cognate subjects insert the factor $(-1)^{\frac{1}{2}m(m-1)}$, and some insert also the factor $1/m^m$.

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The condition is necessary. For, if the cubic has a linear and an irreducible quadratic factor, its roots are

$$y_1 \equiv y_1^{p^n}, \quad y_2, \quad y_3 \equiv y_2^{p^n} \quad (y_2^{p^{2n}} = y_2).$$

Then its discriminant Δ equals π^2, where

$$\pi = (y_1 - y_2)(y_1 - y_2^{p^n})(y_2 - y_2^{p^n}), \quad \pi^{p^n} = -\pi.$$

Since π is not in the $GF[p^n]$, $p > 2$, Δ is a not-square.

The condition is sufficient. For, if Δ is a not-square, the cubic is reducible (§ 1). But not all three roots lie in the $GF[p^n]$, since Δ would then be a square in the field.

3. THEOREM.* *The necessary and sufficient conditions that*

$$x^3 + \beta x + b = 0 \tag{2}$$

be irreducible in the $GF[p^n]$, $p > 3$, *are the following two:*

(3) $R \equiv -4\beta^3 - 27b^2 =$ *a square* $\neq 0$ *in* $GF[p^n]$, *say* $R = 81\mu^2$;

(4) $\frac{1}{2}(-b + \mu\sqrt{-3}) =$ *a not-cube in the field* $(GF[p^n], \sqrt{-3})$.

We suppose that the necessary condition (3) is satisfied. In (2) set $x = y - \beta/3y$. Then

$$(2y^3 + b)^2 = -R/27 = -3\mu^2. \tag{5}$$

Now -3 is a square or a not-square in the $GF[p^n]$, $p > 3$, according as $(x^3 - 1)/(x - 1) = 0$ is or is not solvable, *i. e.*, according as $p^n - 1$ is or is not divisible by 3.

First, let p^n be of the form $3l + 1$, so that -3 is the square of a mark τ in the $GF[p^n]$. Then (5) gives $y^3 = \frac{1}{2}(-b + \mu\tau)$. If the second member were a cube in the field, (2) would have a root in the field. Hence (4) is a necessary condition for irreducibility. Further, it is a sufficient condition. For, if $\frac{1}{2}(-b + \mu\tau)$ is a not-cube ν, $y^3 = \nu$ is irreducible in the $GF[p^n]$. The same is true of (2). Indeed, if it had a root x in the field, then y would satisfy the quadratic $x = y - \beta y^2/3\nu$, in contradiction to the irreducibility of $y^3 = \nu$.

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* From §§ 2, 3, we deduce complete criteria for the nature of the roots of a cubic.

Next, let p^n be of the form $3l+2$, so that $\tau^2=-3$ defines the $GF[p^{2n}]$. If

$$\tfrac{1}{2}(-b+\mu\tau)=(r+s\tau)^3, \quad r \text{ and } s \text{ in the } GF[p^n],$$

then

$$\tfrac{1}{2}(-b-\mu\tau)=(r-s\tau)^3.$$

The product of the left members equals $-\beta^3/27$. Hence

$$y=r+s\tau, \quad -\beta/3y=r-s\tau, \quad x=2r,$$

so that (2) would be reducible. To show that (4) is also a sufficient condition, let $\frac{1}{2}(-b+\mu\tau)$ be a not-cube ν in the $GF[p^{2n}]$. Then if (2) had a root x in the $GF[p^{2n}]$, y would satisfy two equations with coefficients in this field, $x=y-\beta y^2/3\nu$, $y^3=\nu$, whereas the latter is irreducible.

4. When $p^n=3l+2$, there are $\frac{2}{3}(p^{2n}-1)$ not-cubes* ν in the $GF[p^{2n}]$. For a given ν, marks b and β of the $GF[p^n]$ are uniquely determined by the condition $\frac{1}{2}(-b+\mu\sqrt{-3})=\nu$, viz.,

$$-b=\nu+\nu^{p^n}, \quad -\beta^3/27=\nu^{p^n+1}.$$

By the latter, $\beta=-3f\nu^{l+1}$, where $f^3=1$. Then $\beta^{p^n}=\beta$ if and only if

$$f\nu^{\frac{1}{3}(p^{2n}-1)}=1, \quad \beta=-3\nu^{-\frac{1}{3}(p^n-2)(p^n+1)}.$$

The values of b and β are unaltered by the replacement of ν by either ν or ν^{p^n}, but at least one is altered by any new replacement. *When $p^n=3l+2$, there exist exactly $\frac{1}{3}(p^{2n}-1)$ irreducible cubics (2) in the $GF[p^n]$; these are given by*

$$(6) \quad x^3-3x\nu^{-\frac{1}{3}(p^n-2)(p^n+1)}-\nu-\nu^{p^n}=0 \quad (\nu \textit{ a not-cube in } GF[p^{2n}]).$$

5. When $p^n=3l+1$, (4) requires that

$$\tfrac{1}{2}(-b+\mu\sqrt{-3})=\nu, \quad \tfrac{1}{2}(b+\mu\sqrt{-3})=\beta^3/27\nu$$
$$(\nu \text{ a not-cube}),$$

since the product of the first members is $\beta^3/27$. Hence

$$b=\beta^3/27\nu-\nu, \quad \mu\sqrt{-3}=\beta^3/27\nu+\nu.$$

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* The powers of a primitive root with exponents prime to 3.

For these values, (3) is satisfied. Now b is unaltered by the replacement of ν by either ν or $-\beta^3/27\nu$, but is altered by any new replacement. Separating the cases $\beta \neq 0$, $\beta = 0$, we obtain

$$\tfrac{1}{2}\cdot\tfrac{2}{3}(p^n-1)(p^n-1)+\tfrac{2}{3}(p^n-1)\equiv\tfrac{1}{3}(p^{2n}-1)$$

sets ν, β giving distinct sets b, β. *When* $p^n = 3l+1$, *there exist exactly* $\frac{1}{3}(p^{2n}-1)$ *irreducible cubics* (2) *in the* $GF[p^n]$; *these are given by*

$$(7)\qquad x^3+\beta x+\beta^3/27\nu-\nu=0 \qquad (\beta\ \textit{arbitrary},\ \nu\ \textit{not-cube}).$$

6. We may otherwise determine the number of irreducible cubics in the $GF[p^n]$. We enumerate the reducible cubics (2).

First, let there be three linear factors $x-f_j$, where $f_1+f_2+f_3=0$. Consider in turn the cases: (*i*) every $f_j=0$; (*ii*) a single $f_j=0$; (*iii*) each $f_j\neq 0$, two equal; (*iv*) each $f_j\neq 0$, all distinct. For the first three cases, the number of cubics is

$$1+\tfrac{1}{2}(p^n-1)+(p^n-1) \text{ if } p>2;\ 1+(p^n-1)+0 \text{ if } p=2.$$

For case (*iv*) it suffices to take as f_1 any mark $\neq 0$, for f_2 any mark distinct from $0, f_1, -f_1, -2f_1, -\frac{1}{2}f_1$, the last three not occurring if $p=2$, and the last two being superfluous if $p=3$. The number of cubics for case (*iv*) is thus

$$\tfrac{1}{6}(p^n-1)(p^n-5) \text{ if } p>3,\quad \tfrac{1}{6}(p^n-1)(p^n-3) \text{ if } p=3,$$
$$\tfrac{1}{6}(p^n-1)(p^n-2) \text{ if } p=2.$$

The number of cubics for cases (*i*)–(*iv*) is therefore

$$1+\tfrac{1}{6}(p^n-1)(p^n+4) \text{ if } p\neq 3,\quad 1+\tfrac{1}{6}(p^n-1)(p^n+6) \text{ if } p=3.$$

Finally, let the function be $(x-f)(x^2+fx+g)$, where the last factor is irreducible. The number of these functions is $\frac{1}{2}p^n(p^n-1)$, since, of the reducible quadratics, p^n have equal roots and $\frac{1}{2}p^n(p^n-1)$ have distinct roots.

Hence the total number of reducible cubics (2) is

$$\tfrac{1}{3}(2p^{2n}+1) \text{ if } p\neq 3;\quad \tfrac{1}{3}(2p^{2n}+p^n) \text{ if } p=3.$$

But the total number of cubics (2) is p^{2n}. *Hence the number of irreducible functions* $x^3+\beta x+b$ *in the* $GF[p^n]$ *is* $\frac{1}{3}(p^{2n}-1)$ *if* $p\neq 3$, *and* $\frac{1}{3}(3^{2n}-3^n)$ *if* $p=3$.

7. It remains to exhibit the $\frac{1}{3}(3^{2n}-3^n)$ irreducible cubics (2) in the $GF[3^n]$. It is known* that

$$x^p - x/\lambda^{p-1} - \beta/\lambda^p \tag{8}$$

is irreducible in the $GF[p^n]$ if, and only if,

$$\beta^{p^{n-1}} + \beta^{p^{n-2}} + \cdots + \beta^p + \beta \neq 0. \tag{9}$$

Now if $\lambda^{p-1} = \lambda_1^{\,p-1}$, $\beta/\lambda^p = \beta_1\lambda_1^{\,p}$, then $\lambda_1 = \epsilon\lambda$, $\beta_1 = \epsilon\beta$, where $\epsilon^{p-1} = 1$. But if β satisfies condition (9), then also $\epsilon\beta$ does. Hence the number of distinct irreducible functions (8) is

$$(p^n-1)(p^n-p^{n-1}) \div (p-1) \equiv p^{2n-1} - p^{n-1}.$$

Hence for $p = 3$ every irreducible cubic (2) is given by (8), (9).

8. Consider next the quartic

$$x^4 + ax^3 + bx^2 + cx + d = 0. \tag{10}$$

Its discriminant equals that of the resolvent cubic

$$y^3 - by^2 + (ac-4d)y + 4bd - a^2d - c^2 = 0. \tag{11}$$

When (10) is irreducible in the $GF[p^n]$, its discriminant is a not-square (§1), so that (11) is reducible. To verify this result, let

$$\lambda,\quad \lambda^{p^n},\quad \lambda^{p^{2n}},\quad \lambda^{p^{3n}} \qquad (\lambda^{p^{4n}} = \lambda)$$

be the roots of (10). Then (11) has the roots

$$y_1 = \lambda^{p^{2n}+1} + \lambda^{p^{3n}+p^n},\ y_2 = \lambda^{p^n+1} + \lambda^{p^{3n}+p^{2n}},\ y_3 = \lambda^{p^{3n}+1} + \lambda^{p^{2n}+p^n}.$$

Hence $y_1^{\,p^n} = y_1$, $y_2^{\,p^n} = y_3$, $y_3^{\,p^n} = y_2$. But $y_2 \neq y_3$. *Hence when (10) is irreducible in the $GF[p^n]$, its resolvent cubic has a single root in the field.*

Now (11) may be written in the form

$$(\tfrac{1}{4}y^2 - d)(a^2 - 4b + 4y) = (\tfrac{1}{2}ay - c)^2. \tag{11'}$$

But $a^2 - 4b + 4y_1 = t^2$, where

$$t = \lambda + \lambda^{p^{2n}} - \lambda^{p^n} - \lambda^{p^{3n}},\quad t^{p^n} = -t.$$

Hence t is not in the $GF[p^n]$, $p > 2$, unless $t = 0$.

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* Dickson, Linear Groups, p. 29.

9. We first dispose of the special case $t = 0$. Then, by (11′),

$$a^2 - 4b + 4y_1 = 0, \quad \tfrac{1}{2}ay_1 = c.$$

Eliminating y_1, we get

$$c = \tfrac{1}{2}ab - \tfrac{1}{8}a^3. \tag{12}$$

Then the quartic (10) may be written in the form

$$(x^2 + \tfrac{1}{2}ax + \tfrac{1}{2}b - \tfrac{1}{8}a^2)^2 = k, \quad k \equiv (\tfrac{1}{2}b - \tfrac{1}{8}a^2)^2 - d. \tag{13}$$

Then k must be a not-square; otherwise, the quartic would have two quadratic factors with coefficients in the $GF[p^n]$.

Suppose that this condition on k is satisfied. In view of (13), the quartic has no root in the $GF[p^n]$. It remains to find the further conditions in order that (10) shall not be the product of two quadratic factors in the field. Such factors, when existing, must be of the form

$$x^2 + (\tfrac{1}{2}a + r)x + s, \quad x^2 + (\tfrac{1}{2}a - r)x + ds^{-1},$$

where

$$s + ds^{-1} + \tfrac{1}{4}a^2 - r^2 = b, \quad s(\tfrac{1}{2}a - r) + ds^{-1}(\tfrac{1}{2}a + r) = c. \tag{14}$$

Multiplying the first by $\frac{1}{2}a$ and subtracting the result from the second, we obtain, in view of (12),

$$r(ds^{-1} - s + \tfrac{1}{2}ar) = 0. \tag{15}$$

If $r = 0$, (14_1) becomes $(2s + \frac{1}{4}a^2 - b)^2 = 4k$. Hence $r \neq 0$. If $a = 0$, then $s^2 = d$. If d is a square, (14_1) is satisfied when r^2 equals $\pm 2\sqrt{d} - b$; but one of these values is a square, since their product equals the not-square $4k$. The result for this case is stated in the corollary below. Let next $a \neq 0$. If we eliminate r between (15) and (14_1), we obtain after simple modifications

$$\left\{\frac{2}{a}\left(s + \frac{d}{s}\right) - \frac{a}{4}\right\}^2 = w, \qquad w \equiv \frac{5}{16}a^2 - b + 16d/a^2.$$

There exist solutions s in the $GF[p^n]$ if and only if w is a square. In fact, when w is a square, s lies in the field if

$$(\tfrac{1}{8}a^2 \pm \tfrac{1}{2}aw^{\frac{1}{2}})^2 - 4d$$

is a square. But the product of these two expressions is seen to equal the not-square $\frac{1}{4}a^4k$, so that one is a square.

THEOREM. *If the coefficients of the quartic* (10) *satisfy* (12), *it is irreducible in the* $GF[p^n]$, $p > 2$, *if and only if* $(\frac{1}{2}b - \frac{1}{8}a^2)^2 - d$ *and* $\frac{5}{16}a^4 - a^2b + 16d$ *are not-squares.*

Corollary. *In the* $GF[p^n]$, $p > 2$, $x^4 + bx^2 + d$ *is irreducible if and only if* $b^2 - 4d$ *and* d *are not-squares.*

10. THEOREM. *The necessary and sufficient conditions that a quartic* (10), *not satisfying* (12), *be irreducible in the* $GF[p^n]$, $p > 2$, *are that the resolvent cubic shall have* one and but one root* y_1 *in the* $GF[p^n]$ *and that* $a^2 - 4b + 4y_1$ *shall be a not-square.*

These conditions are necessary by § 8 since now $t \neq 0$.

It remains to prove that these conditions are sufficient. Since there are three ways of separating the four roots into pairs, there are exactly three sets of two quadratic factors of (10), viz.,

$$x^2 + \tfrac{1}{2}(a \mp \tfrac{1}{2}t_\iota)x + \tfrac{1}{2}y_\iota \mp (\tfrac{1}{2}ay_\iota - c)/t_\iota = 0 \quad (t_\iota^2 = y_\iota).$$

Note that under the assumptions each $t_\iota \neq 0$. If a pair of factors have as coefficients marks of the $GF[p^n]$, then y_ι and t_ι belong to the field. But y_2, y_3 and t_1 are not in the $GF[p^n]$. Hence there is no decomposition into quadratic factors in the field. Finally, we show that there is no linear factor. The quartic may be written in the form

$$X^2 = (a^2 - 4b + 4y_1)Z^2, \quad X \equiv x^2 + \tfrac{1}{2}ax + \tfrac{1}{2}y_1,$$

$$Y \equiv \tfrac{1}{2}x + \frac{\tfrac{1}{2}ay_1 - c}{a^2 - 4b + 4y_1}.$$

If the quartic has a root x in the field, then $X = Z = 0$. To prove that this is impossible it suffices to set $a = 0$, as may be accomplished by a transformation of the quartic. Then

$$x^2 + \tfrac{1}{2}y_1 = 0, \quad \tfrac{1}{2}x - c/(4y_1 - 4b) = 0$$

require that

$$y_1^3 - 2by_1^2 + b^2y_1 + \tfrac{1}{2}c^2 = 0. \tag{16}$$

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* We may replace this condition by the condition that Δ shall be a not-square (§ 2).

On eliminating first y_1^3 and then c^2 between (16) and (11), and in the latter case removing the factor $y_1 - b \neq 0$, we get

$$(17)\quad by_1^2 - b^2y_1 - 4dy_1 + 4bd - \tfrac{3}{2}c^2 = 0, \quad 3y_1^2 - 2by_1 - 4d = 0.$$

But the Sylvester eliminant of these is found to equal $-\frac{3}{4}\Delta$, where Δ is the discriminant of (11) for $a = 0$. Hence the case $p \neq 3$ is excluded. For $p = 3$, $by_1 = d \neq 0$; but for this value of y_1 (11) becomes $\Delta = 0$. Hence in every case the conditions are sufficient to make the quartic irreducible.

UNIVERSITY OF CHICAGO,
June, 1906

ON THE THEORY OF EQUATIONS IN A MODULAR FIELD.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, September 3, 1906.)

1. The object of this note is to point out that the Galois theory of algebraic equations may be extended to equations in a field F having a modulus p. For a finite field the theory is so obvious that this case furnishes a simple, but characteristic, example of the Galois theory.

2. Let the domain of rationality be the $GF[p^n]$. Consider an equation $f(x) = 0$ with coefficients in this field and having distinct roots $x_1, \cdots, x_m$. By a Galois resolvent of $f(x) = 0$ will be meant an equation $\phi(V) = 0$, irreducible in the $GF[p^n]$, with a root V which is a rational function of $x_1, \cdots, x_m$, and such that each x_i is a rational function of V with coefficients in the field. Let $m_1, m_2, \cdots$ be the distinct degrees of the irreducible factors of $f(x)$ in the $GF[p^n]$, and let l be the least common multiple of $m_1, m_2, \cdots$. Then the smallest field which contains all the roots x_i of $f(x) = 0$ is the $GF[p^{nl}]$. Any primitive root of the latter may be taken as the desired function V. In fact the power $(p^{nl} - 1)/(p^{nm_i} - 1)$ of V is a primitive root ρ_i of the $GF[p^{nm_i}]$, so that the x's are powers of $\rho_1, \rho_2, \cdots$ and hence of V. Further, V is a rational function of the ρ's and hence of the x's. Indeed, if m_1 and m_2 have the greatest common divisor g and the least common multiple λ, an

irreducible equation of degree m_1 decomposes in the $GF[p^{nm_2}]$ into g irreducible equations, any one of which defines the $GF[p^{n\lambda}]$, so that the marks of the latter are rational functions of ρ_1 and ρ_2.

The Galois resolvent $\phi(V) = 0$ has the m roots

$$V^{p^{ni}} \qquad (i = 0, 1, \cdots, m-1).$$

Hence the Galois group of $f(x) = 0$ in the $GF[p^n]$ is simply isomorphic with the cyclic group of order m generated by the substitution

$$V' = V^{p^n}.$$

3. One standard method (following Galois) of obtaining the Galois resolvent of an algebraic equation of degree m in a domain of rationality R depends upon the existence of an $m!$ valued rational function V of the roots x_i. When R is the $GF[p^n]$, this method cannot be applied unless p^n exceeds a certain limit. For example, if all the x's belong to the $GF[p^n]$, V evidently does not exist when $m! > p^n$. Again, if the irreducible factors of $f(x)$ are all of degree $\leqq 2$, V does not exist when $m! > p^{2n}$. Except for certain values of p^n, small in relation to m, there exists an $m!$ valued function V and the theory presents no difficulty.*

4. There is an interesting point in the determination of an $m!$ valued linear function of the distinct roots $x_1, \cdots, x_m$. It is illustrated in the case $m = 3$. If $t = \alpha x_1 + \beta x_2 + \gamma x_3$ is six valued, so is $t - \gamma(x_1 + x_2 + x_3)$ six valued. Hence we may restrict our attention to $V = qx_1 + x_2$. The necessary and sufficient condition that V be six valued is that q be distinct from

$$0,\ 1,\ (x_2 - x_3)/(x_1 - x_3),\ (x_1 - x_2)/(x_1 - x_3),\ (x_3 - x_2)/(x_1 - x_2),$$

and the reciprocals of the last three. Denoting the first of these fractions by λ, we find that the other two are $1 - \lambda$ and

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* In a discussion of the Galois group of order N (*cf.* Dickson's Algebraic Equations, p. 53), there is a proof that if $\phi_1 = \phi_2 = \cdots = \phi_N$, then $\phi_1 = N^{-1}(\phi_1 + \cdots + \phi_N) = N^{-1}$ (a number of R). This step would now be invalid if N is a multiple of the modulus p. But we may proceed as follows: Let $N = p^s k$, k prime to p. Take the ${}_N C_p = p^{s-1}k_1$ products π_i of the ϕ's p at a time; take the $p^{s-2}k_2$ products π'_i of the π_i p at a time; etc. Finally, we reach $k_s \phi^{p^s}$ as a symmetric function of the ϕ_i. But k_s is prime to p, and the extraction of (p^s)th roots is possible in the field.

$\lambda/(\lambda-1)$. Hence

$$(1)\quad q \neq 0,\ 1,\ \lambda,\ 1-\lambda,\ \frac{\lambda}{\lambda-1},\ \frac{1}{\lambda},\ \frac{1}{1-\lambda},\ \frac{\lambda-1}{\lambda}\quad (\lambda \neq 0,\ 1).$$

Here the six functions of λ are the six elements of the cross-ratio group, and each differs from 0 and 1. Hence equalities arise only when $\lambda = -1$, 2, or $\frac{1}{2}$, or when $\lambda^2 - \lambda + 1 = 0$.

When x_1, x_2, x_3 *are distinct,* $qx_1 + x_2$ *is six valued only in the following cases:* (*i*) *one of the* x*'s is an arithmetical mean between the other two, with* $q \neq 0, 1, -1, 2, \frac{1}{2}$; (*ii*) $\Sigma x_j^2 = \Sigma x_j x_k$, *with* $q \neq 0, 1, \lambda, 1/\lambda$ (*where* $\lambda^2 - \lambda + 1 = 0$); (*iii*) *neither of the relations on the* x*'s holding, with* q *not equal to one of the eight distinct values* (1).

It may now be readily shown that there exist six valued linear functions of the roots x_i of a cubic in the $GF[p^n]$ when $p^n > 8$; when $p^n = 7$; and when $p^n = 5$ or 8, with the x_i not all in the $GF[p^n]$.

5. In conclusion it may be remarked that the Galois theory as presented in Weber's Algebra may readily be extended to apply to modular fields, provided his argument on page 500 (of volume 1 of the second edition) be replaced by that in § 2 above.

THE UNIVERSITY OF CHICAGO,
July, 1906.

NOTE ON THE VARIATION OF THE DEFINITE INTEGRAL.

BY MR. N. J. LENNES.

(Read before the Chicago Section of the American Mathematical Society. April 14, 1906.)

A function is said to be of limited variation on an interval ab if the set of sums

$$\left[\sum_{i=0}^{n-1} |f(x_i) - f(x_{i+1})|\right]$$

is bounded for the set of all partitions of ab. The points $a = x_0, x_1, x_2 \cdots, x_{n-1}, x_n = b$ of each partition are ordered on the interval according to the subscripts. The least upper bound of this set of sums is denoted by $V_a^b f(x)$ and is called the variation of $f(x)$ on ab. We consider the variation of

$$\int_a^x f(x)dx$$

on an interval ab, where

$$\int_a^x f(x)dx$$

is a function of the upper limit of integration. Denote this variation by

$$V_a^b \int_a^x f(x)dx.$$

THEOREM I. *If $f(x)$ does not change sign on ab then*

$$V_a^b \int_a^x f(x)dx$$

exists on ab and

$$V_a^b \int_a^x f(x)dx = \int_a^b |f(x)|\, dx.$$

Proof: The theorem is obvious since for every partition of ab consisting of $n + 1$ points

$$\sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(x)dx \right| = \int_a^b |f(x)|\, dx.$$

For convenience of reference a definition of the definite integral is inserted. Let ab be an interval upon which a function $f(x)$ is defined, single valued and bounded. Let π_δ stand for any partition of ab consisting of the points $x_0 = a, x_1, \cdots, x_{n-1}, x_n = b$ such that $\Delta_1 x = x_1 - a$, $\Delta_2 x = x_2 - x_1$, $\cdots$, $\Delta_n x = b - x_{n-1}$, each interval being numerically less than or equal to δ. Let $\xi_1, \xi_2, \cdots, \xi_n$ be any points of the intervals $x_0 x_1$, $x_1 x_2$, $\cdots$, $x_{n-1} x_n$ respectively and let

$$S_\delta = f(\xi_1)\Delta_1 x + f(\xi_2)\Delta_2 x + \cdots + f(\xi_n)\Delta_n x = \sum_{i=1}^{n} f(\xi_i)\Delta_i x.$$

If the many valued function S_δ of δ approaches a single limiting value as δ approaches zero, then

$$\mathop{\mathrm{L}}_{\delta=0} S_\delta = \int_a^b f(x)dx.$$

$S_a^b\delta$ represents the sum S_δ over the interval ab.

LEMMA I. *The function S_δ in the definition of the definite integral approaches its limit uniformly with respect to x, i. e., for every ϵ there exists a δ_ϵ such that*

$$\left| \int_{x_1}^{x_2} f'(x)dx - S_{x1}^{x_2}\delta_\epsilon \right| < \epsilon$$

for every $S_{x1}^{x_2}\delta_\epsilon$ and for every pair of points x_1 and x_2 on ab.

Proof: Since by hypothesis

$$\mathop{\mathrm{L}}_{\delta=0} S_a^b\delta = \int_a^b f(x)dx,$$

here exists for a given ϵ a $\delta_{\epsilon/2}$ such that

$$(1) \qquad \left| \int_a^b f(x)dx - S_a^b\delta_{\epsilon/2} \right| < \frac{\epsilon}{2}$$

for every $S_a^b\delta_{\epsilon/2}$. Then for any pair of points x_1, x_2 on ab

$$(2) \qquad \left| \int_a^{x_1} f(x)dx - S_a^{x_1}\delta_{\epsilon/2} + \int_{x_1}^{x_2} f(x)dx - S_{x1}^{x_2}\delta_{\epsilon/2} \right.$$
$$\left. + \int_{x_2}^b f(x)dx - S_{x_2}^b\delta_{\epsilon/2} \right| < \frac{\epsilon}{2}.$$

Since $f(x)$ is integrable on ax_1 and x_2b, it follows that there exists a $\delta'_{\epsilon/2}$ for ax_1 and a $\delta''_{\epsilon/2}$ for x_2b such that

$$(3) \qquad \left| \int_a^{x_1} f(x)dx - S_a^{x_1}\delta'_{\epsilon/2} + \int_{x_2}^b f(x)dx - S_{x_2}^b\delta''_{\epsilon/2} \right| < \epsilon/2.$$

Then from (2) and (3)

$$\left| \int_{x_1}^{x_2} f(x)dx - S_{x1}^{x_2}\delta_{\epsilon/2} \right| < \epsilon.$$

Hence $\delta_{\epsilon/2}$ is the δ_ϵ required by the lemma.*

Definition: The difference between the least upper and the greatest lower bound of a function on an interval is the oscillation of the function on that interval.

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*If δ_ϵ 2 is so chosen that $\delta_{\epsilon/2} \leqq \delta'_{\epsilon/2}$ and $\delta_{\epsilon/2} \leqq \delta''_{\epsilon/2}$, the argument is a little more obvious, though this restriction is not necessary.

LEMMA II. *If the oscillation of a function on an interval ab is less than* $\epsilon/2$ *then for any two partitions* π *and* π' *of ab*

$$\left|\sum_{i=1}^{n}|\Delta_i x|\cdot|f(\xi_i)|-\left|\sum_{i=1}^{n'}\Delta'_i x f(\xi'_i)\right|\right|<\epsilon|b-a|$$

when n and n' are the numbers of intervals on the partitions π *and* π' *respectively.*

Proof: Let B be any value of $f(x)$ on ab. Then for any interval $x_i x_{i+1}$ of the partition π

$$\left||f(\xi_i)|\cdot|\Delta_i x|-|\beta|\cdot|\Delta_i x|\right|<\frac{\epsilon}{2}\,\Delta_i x.$$

Hence

$$(1)\qquad \left|\sum_{i=1}^{n}|\Delta_i x|\cdot|f(\xi_i)|-|\beta|\cdot|b-a|\right|<\frac{\epsilon}{2}|b-a|.$$

Similarly for any interval $x'_i x'_{i+1}$ of the partition π'

$$|f(\xi'_i)\,\Delta'_i x-\beta\cdot\Delta'_i x|<\frac{\epsilon}{2}|\Delta'_i x|.$$

Hence

$$\left|\sum_{i=1}^{n'}f(\xi'_i)\Delta'_i x-\beta\cdot(b-a)\right|<\frac{\epsilon}{2}|b-a|$$

and

$$(2)\qquad \left|\left|\sum_{i=1}^{n'}f(\xi'_i)\Delta'_i x\right|-|\beta|\;|b-a|\right|<\frac{\epsilon}{2}|b-a|.$$

From (1) and (2) we have

$$\left|\sum_{i=1}^{n}|\Delta_i x|\,|f(\xi_i)|-\left|\sum_{i=1}^{n'}\Delta'_i x\cdot f(\xi'_i)\right|\right|<\epsilon|b-a|.$$

THEOREM II. If $\int_a^b f(x)dx$ *exists, then* $V_a^b\int_a^x f(x)dx$ *exists and*

$$V_a^b\int_a^x f(x)dx=\int_a^b|f(x)|\,dx.$$

Proof: Since for any two values of x, x_1 and x_2 on ab

$$(1)\qquad \left|\int_{x_1}^{x_2}f(x)dx\right|\leqq\int_{x_1}^{x_2}|f(x)|dx$$

and since by a well-known theorem

$$\int_{x_1}^{x_2} |f(x)|\, dx$$

exists if

$$\int_{x_1}^{x_2} f(x)dx$$

exists, it follows from (1) and Theorem I that

$$V_a^b \int_a^x f(x)dx$$

exists and that

$$V_a^b \int_a^x f(x)dx \leqq V_a^b \int_a^x |f(x)|\, dx. \tag{2}$$

It remains to show that the sign $<$ in (2) is impossible. Since $f(x)$ is integrable on ab it follows by a well known theorem that for any pair of positive numbers σ and λ there exists a partition π such that the sum of the intervals on which the oscillation of $f(x)$ is greater than σ is less than λ.

For a preassigned ϵ let $\lambda = \epsilon/M$, where M is the difference between the least upper and the greatest lower bound of $f(x)$ on ab. Let $\sigma = \epsilon/4|b - a|$. Denote by I the set of intervals on ab on which the oscillation is less than σ and by I' the complementary set of intervals. Let n be the number of intervals in I. By Lemma I there exists a δ such that

$$\left| \int_{x_1}^{x_2} f(x)dx - S_{x_1}^{x_2}\delta \right| < \frac{\epsilon}{2n} \tag{3}$$

for every pair of points x_1 and x_2 on ab and for every $S_{x_1}^{x_2}\delta$. Let x_1, x_2 be the extremities of a segment of the set I. By Lemma II

$$\left| |S_{x_1}^{x_2}\delta| - |x_1 - x_2| \cdot |f(\xi_2)| \right| < \frac{\epsilon}{2|b - a|}\, |x_1 - x_2|. \tag{4}$$

Denoting the integrals on the segments of the set I by

$$\int_i f(x)dx,$$

the lengths of the segments by Δ_i, and the sums S_δ on the segments of I by $S_{x_j}^{x_{j+1}}$, we have from (4)

$$\left|\sum_{j=1}^{j=2n-1} |S_{x_j}^{x_{j+1}}\delta| - \sum_{i=1}^{n} \Delta_i |f(\xi_i)|\right| < \frac{\epsilon}{2}$$

where $2n - 1$ is the number of extremities of the segments of I and where j takes only odd values.

Then from (3)

$$\left|\sum_{j=1}^{2n-1} |S_{x_j}^{x_{j+1}}\delta| - \sum_{i=1}^{n}\left|\int_i f(x)dx\right|\right| < \frac{\epsilon}{2},$$

since if

$$\left|\int_{x_1}^{x_2} f(x)dx - S_{x_1}^{x_2}\delta\right| < \frac{\epsilon}{2n},$$

then

$$\left|\left|\int_{x_1}^{x_2} f(x)dx\right| - |S_{x_1}^{x_2}\delta|\right| < \frac{\epsilon}{2n}.$$

Hence

$$\left|\sum_{i=1}^{n} \Delta_i |f(\xi_i)| - \sum_{i=1}^{n}\left|\int_i f(x)dx\right|\right| < \epsilon.$$

Since

$$\underset{\delta=0}{\mathrm{L}}\sum_{j=1}^{n'} \Delta_j |f(\xi_j)| = \int_i |f(x)|dx$$

and since ϵ is arbitrary it is evident that the least upper bound of

$$\sum_{j=1}^{n'}\left|\int_j f(x)dx\right|$$

for the set of all partitions of I cannot be less than

$$\sum_{i=1}^{n}\int_i |f(x)|dx$$

(where

$$\int_j f(x)dx$$

denotes the integral on an interval formed by partitioning an interval of I.)

Since the sum of the lengths of the interval on which the oscillation is greater than σ is less than $\lambda = \epsilon / M$ and since by a known theorem

$$\sum_{i=1}^{n''} \int_{i'} |f(x)| dx < \lambda \; M = \epsilon$$

(where

$$\int_{i'} f|(x) dx$$

denotes the integral on a segment of I') it follows from the arbitrary character of ϵ that the least upper bound of

$$\left[\sum_{i=0}^{n} \left| \int_{x_i}^{x_{i+1}} f(x) dx \right| \right]$$

for the set of all partitions of ab cannot be less than

$$\int_a^b |f(x)| \, dx,$$

which proves the theorem.

Definition: The length of a curve represented by the equation $y = f(x)$ on an interval ab is the least upper bound of the set of sums

$$\left[\sum_{i=1}^{n} \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2} \right]$$

for the set of all partitions $a = x_1, x_2, \cdots, x_{n-1}, x_n = b$ of the interval ab, where y_i corresponds to x_i by the functional correspondence $y = f(x)$.

THEOREM III. *If on the interval ab and on every subinterval of ab the functions* $f_1(x)$ *and* $f_2(x)$ *are of equal variation then the curves represented by* $y = f_1(x)$, *and* $y = f_2(x)$ *are of equal length on the interval* ab.

Proof: Consider any partition π of ab consisting of the points $x_i (i = 0, \cdots, n)$. Then

$$\sum_{i=0}^{y} \sqrt{(x_i - x_{i+1})^2 + (f(x_i) - f(x_{i+1}))^2}$$

is one sum of the set of sums whose least upper bound is the length of the curve $y = f(x)$ on ab.

By hypothesis $f_1(x)$ and $f_2(x)$ are of the same variation on any interval $x_\iota x_{r+1}$ of π. Obviously the variation of $f_1(x)$ on $x_\iota\, x_{\iota+1}$ is equal to or greater than $|f_1(x_\iota) - f(x_{\iota+1})|$. Let ϵ be any preassigned positive number and let $\epsilon' = \epsilon/n$ where n is the number of interval in the partition π. Then there is a partition π'_ι of $x_\iota\, x_{\iota+1}$ consisting of the points $x'_0 = x_\iota, x'_2, \cdots, x'_j, \cdots, x'_{n'} = x_{\iota+1}$ such that

$$(1) \qquad |f_1(x_\iota) - f_1(x_{\iota+1})| - \sum_{j=0}^{n'} |f_2(x_j) - f_2(x_{j+1})| < \epsilon'.$$

By means of a broken line whose segments are equal in length to the lengths of the segments connecting the points $(x_j, f_2(x_j))$ and $(x_{j+1}, f_2(x_{j+1}))$ connect the point $(x_\iota, f_2(x_\iota))$ with a point $(x_{\iota+1}, y_1)$ on the line $x = x_{\iota+1}$ in such manner that the slopes of these segments are, say, all positive. Then by (1)

$$(2) \qquad |f_1(x_\iota) - f_1(x_{\iota+1})| - |f_2(x_1) - y_1| < \epsilon'.$$

Hence obviously

$$(3) \quad \sqrt{(x_\iota - x_{\iota+1})^2 + [f_1(x_\iota) - f_1(x_{\iota+1})]^2}$$
$$- \sum_{j=0}^{n'} \sqrt{(x_j - x_{j+1})^2 + [f_2(x_j) - f_2(x_{j+1})]^2} < \epsilon' \cdots.$$

Let π'', consisting of the points $x_1 = a, x_2, \cdots, x_k, \cdots, x_{n''} = b$, be the partition of ab containing all the points of the partitions π'_ι of the intervals $x_\iota x_{\iota+1}$, and let n'' be the number of intervals in this partition. Then from (3)

$$(4) \quad \sum_{\iota=0}^{n} \sqrt{(x_\iota - x_{\iota+1})^2 + [f_1(x_\iota) - f_1(x_{\iota+1})]^2}$$
$$- \sum_{k=0}^{n''} \sqrt{(x_k - x_{k+1})^2 + [f_2(x_k) - f_2(x_{k+1})]^2} < \epsilon' n = \epsilon \cdots.$$

By a well known theorem the lengths on ab of the curves $y = f_1(x)$ and $y = f_2(x)$ both exist, since the functions are of limited variation. Moreover, it follows from (4) that the length of the curve $y = f_2(x)$ cannot be less than that of $y = f_1(x)$, ϵ being arbitrary. In the same manner it is shown that the length of the curve $y = f_1(x)$ cannot be less than the length of the curve $y = f_2(x)$. Hence the curves are of equal length and the theorem is proved.

Theorems II and III yield the following interesting
COROLLARY. *If $f(x)$ is such that*

$$\int_a^x f'(x)dx = f(x) - f(a)$$

on ab then the length on ab of the curve $y = f(x)$ can be found by finding the length of the curve

$$y = \int_a^x |f'(x)|\, dx$$

on this interval.

Proof: By Theorem II the curves

$$y = \int_a^x f'(x)dx = f(x) - f(a) \quad \text{and} \quad y = \int_a^x |f'(x)|\, dx$$

have the same variation on ab and on every subinterval of ab, whence by theorem III their lengths are equal.

This corollary reduces the problem of finding the length of a curve of the class specified in the corollary to the finding of the length of a non-oscillating curve.

We now use theorem II to give a fresh proof of a theorem on improper definite integrals.

Definition: A function is integrable at a point x_0 if there exists an interval containing x_0 as an interior point on which $f(x)$ is properly integrable. (A function is properly integrable on an interval only in case it is bounded on that interval.)

Consider a function $f(x)$ which is integrable at every point of the interval ab except at a set of points $[P]$ which is of content zero.*

Let $[I]$ be any finite set of non-overlapping intervals on ab such that no point of $[P]$ lies on an interval of $[I]$. Then the integral of $f(x)$ exists properly on every interval of $[I]$. Denote by

$$\int_{aI}^b f(x)dx$$

the sum of the integrals of $f(x)$ on the intervals of $[I]$.

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* A set of points $[P]$ is of content zero if for every positive number ε there exists a finite set of intervals $[I']$ such that every point of $[P]$ lies on at least one interval of $[I']$ and further such that the sum of the lengths of the intervals of $[I']$ is less than ε.

Denote by $m[I]$ the sum of the lengths of these intervals and let $D = |a - b|$.

If the limit

$$\mathop{\mathrm{L}}_{m[I]=D} \int_{aI}^{b} f(x)dx$$

exists and is finite, this limit is said to be the improper definite integral of $f(x)$ on ab.

THEOREM IV. *The improper definite integral regarded as a function of the upper limit of integration is of limited variation on any interval ab.*

Proof: Since by hypothesis the improper definite integral of $f(x)$ exists on ab it follows that there exists an M such that for every set of interval $[I]$

$$(1) \qquad \left| \int_{aI}^{b} f(x)dx \right| < M.$$

If the theorem fails to hold, *i. e.*, if for every M' there is a partition π of ab such that

$$(2) \qquad \sum_{i=0}^{n} \left| \int_{x_i}^{x_{i+1}} f(x)dx \right| > M',$$

then for a certain subset $x_j\, x_{j+1}$ of the set of interval $x_i\, x_{i+1}$

$$(3) \qquad \sum_{j=0}^{n'} \left| \int_{x_j}^{x_{j+1}} f(x)dx \right| > \frac{M'}{2}.$$

(The integrals in (3) are improper definite integrals.) Since the improper definite integral exists on each of the intervals $x_j\, x_{j+1}$ there exists a set of segments $[I']$ such that

$$(4) \qquad \int_{aI'}^{b} f(x)dx > \frac{M'}{2}.$$

If $M' = 2M$ then (4) contradicts (1). Hence the theorem is proved.

THEOREM V. *If the improper definite integral of $f(x)$ exists on the interval ab then the improper definite integral of $|f(x)|$ exists on a b.*

Proof: By theorem IV

$$\int_a^x f(x)dx$$

is of limited variation on ab. Hence there exists a number M such that for every set of intervals $[I]$

$$V_a^b \int_{x_1}^x f(x)dx < M.$$

But by Theorem II,

$$V_a^b \int_{x_1}^x f(x)dx = \int_{aI}^b |f(x)|\,dx < M,$$

which proves the theorem.*

CHICAGO,
July 17, 1906.

A NOTE ON TRANSITIVE GROUPS.

BY DR. W. A. MANNING.

(Read before the American Mathematical Society, September 3, 1906.)

THREE unconnected topics in the theory of transitive substitution groups are touched on in this note.

1.

THEOREM I. *The largest subgroup of a transitive group G of degree n, in which a subgroup H leaving fixed m $(0 < m < n)$ letters is invariant, has as many transitive constituents in these m letters as there are different conjugate sets in G_1 (a subgroup of G, that leaves one of the m letters fixed) which, under the substitutions of G, enter into the complete set of conjugates to which H belongs. Moreover, the degree of each of these constituents is proportional*

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* For a general discussion of the improper definite integral see E. H. Moore, "Concerning Harnack's theory of improper definite integrals" *Transactions Amer. Math. Society*, volume 2, pp. 296–330, and pp 459–475 same volume. See also references given in this paper by Professor Moore. For a proof of Theorem V. of the present note see Jordan, Cours d'analyse, ed. 2, vol. 2 (1894), pp. 46 ff.

to the number of subgroups in the several conjugate sets of G_1 *in question.**

Let g be the order of G, and let those conjugates of H, which are found in G_1, lie in κ different sets, in so far as they are permuted by the substitutions of G_1 only, with c conjugates in the set including H, c_1 in a second set, and so on. In G_1, H is invariant in a subgroup of order g/nc, while H_1, a subgroup in the second set, is invariant in a group of order g/nc_1. The largest subgroup I of G in which H is invariant is of order $gm/n\sigma$ $(\sigma = c + c_1 + \cdots + c_{\kappa-1})$. Now I does not connect transitively the $n - m$ letters displaced by H and the m letters it leaves fixed. Since the largest subgroup of G_1 in which H is invariant is of order g/nc, I has one transitive constituent of degree $(gm/n\sigma) \div (g/nc) = mc/\sigma$ in letters left fixed by H. The group H_1 is transformed into H by some substitution S not in G_1. Then in the subgroup $G_2 = S^{-1}G_1S$, H belongs to a set of g/nc_1 conjugates (conjugate under the substitutions of G_2). Hence I has a transitive constituent of degree mc_1/σ in letters left fixed by H. Continuing thus, we see that I has transitive constituents of degrees $mc/\sigma, mc_1/\sigma, \cdots, mc_{\kappa-1}/\sigma$. These constituents involve all the letters left fixed by H. This completes the proof of the theorem.

COROLLARY. *If* H *is a Sylow subgroup of* G_1, I *has a transitive constituent of degree* m.

2.

THEOREM II. *If the class of a t-ply transitive group* $(t > 3)$ *is not less than* $n - 2t + 5$, *then the degree* n *is less than or equal to* $\frac{1}{2}(t^3 - t + 2) + (t - 1)!$

There are three cases to consider separately:

G contains a substitution of order 2 and (1) of degree $n - t + 2$, (2) of degree $n - t + 1$, (3) of degree $n - t - \epsilon$ $(0 \leqq \epsilon \leqq t - 5)$. Since the subgroup of G that leaves fixed $t - 2$ letters is of even order, it always contains a substitution of order 2 falling under one of the three following categories:

1. If there is one substitution of order 2 on just $n - t + 2$ letters in G, there are at least $[n(n-1)\cdots(n-t+3)(n-t+1)]/(t-2)!$ The total number of transpositions involved in these substitutions is $n(n-1)\cdots(n-t+1)/2(t-2)!$ Of these substitutions at least $n(n-1)\cdots(n-t+1)\,n\,(n-1)$

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* Cf. Burnside, Theory of Groups (1897), p 202, where a special case is proved.

p 17598

$(t-2)\,!$ have some particular transposition as (ab) in common. Again by the same reasoning at least $n(n-1)\cdots(n-t+1)(n-t)/n(n-1)(n-2)(n-3)(t-2)\,!$ that have (ab) in common have also a transposition (cd) in common. Finally the number of these substitutions which have $t-2$ common transpositions is at least

$$(1)\qquad \frac{n(n-1)\cdots(n-t)(n-t-2)\cdots(n-3t+8)}{(t-2)\,!\;n(n-1)\cdots(n-2t+6)(n-2t+5)} \leqq 1.$$

That this number cannot be greater than unity comes from the fact that G is by hypothesis of class $n-2t+5$, while the product of two substitutions with $t-2$ transpositions in common is of degree less than or equal to $n-2t+4$. This inequality is of the first degree in n, so that we obtain from it

$$(2)\qquad n \leqq \tfrac{1}{2}(t^2-3t+4)+(t-2)!.$$

2. In case G has a substitution of order 2 and degree $n-t+1$, we get by the same process

$$(3)\quad \frac{n(n-1)\cdots(n-t+1)(n-t-1)\cdots(n-3t+7)}{(t-1)\,!\,n(n-1)\cdots(n-2t+6)(n-2t+5)} \leqq 1,$$

from which

$$(4)\qquad n \leqq \tfrac{1}{2}(t\ -t-2)+(t-1)\,!$$

When $t=4$, G is of class $n-3$, and the above inequality gives $n \leqq 11$.

3. The inequality in this case is

$$(5)\quad \frac{n(n-1)\cdots(n-t+1)(n-t-\varepsilon)(n-t-\varepsilon-2)\cdots(n-3t-\varepsilon+6)}{(t+\varepsilon)(t+\varepsilon-1)\cdots(\varepsilon+1)(n)(n-1)\cdots(n-2t+3)} \leqq 1.$$

It is of the second degree in n, and gives

$$(6)\qquad n \leqq \tfrac{1}{4}(t^2-3t-8+2t\varepsilon-4\varepsilon)+\sqrt{(t+\varepsilon)\,!/\varepsilon\,!}$$

The limit $\frac{1}{2}(t^3-t+2)+(t-1)\,!$ is always the highest of the three.

In this connection it is easy to establish the following

THEOREM III. *If the subgroup leaving t letters fixed in a t-ply* ($t > 2$) *transitive group G is of even order, the degree of*

G is less than $\frac{1}{2}(t^2 - t + 6) + \varepsilon(t - 1) + (t + \varepsilon)!/\varepsilon!$ *unless the class is less than* $n - 2t + 3$.

The inequality

$$\frac{n(n-1)\cdot(n-t+1)(n-t-\varepsilon)(n-t-\varepsilon-2)\cdots(n-3t-\varepsilon+4)}{n(n-1)\cdots(n-2t+6)(n-2t+5)} \leqq \frac{(t+\varepsilon)!}{\varepsilon!}$$

where $0 \leqq \varepsilon \leqq t - 3$, is found as before, and from it the theorem follows.

3.

Another theorem of value in the applications is the following :

THEOREM IV. *A doubly transitive group cannot contain an invariant imprimitive subgroup unless its degree is a power of a prime. Then the group is a subgroup of the holomorph of the abelian group of order* p^a *and type* $(1, 1, \cdots)$.

On pages 193 and 194 of his Theory of Groups Burnside proves that the invariant imprimitive subgroup H is of degree n and class $n - 1$ and that the $n - 1$ substitutions of degree n in H form a single conjugate set under G. Then by Frobenius's theorem on groups of "class $n - 1$," H contains a characteristic subgroup of degree and order n which is abelian with all its operators of the same order.

STANFORD UNIVERSITY,
June 19, 1906.

DIFFERENTIAL GEOMETRY OF n DIMENSIONAL SPACE.

Sur les Systèmes Triplement Indéterminés et sur les Systèmes Triple-Orthogonaux. Par C. GUICHARD. Scientia, no. 25. Gauthier-Villars, Paris, 1905. viii + 95 pp.

DURING the past ten years the field of differential geometry has been greatly enriched by the researches of M. Guichard. The eminent geometer has made a profound study of the properties of ordinary space by means of operations defined for space of n dimensions. He has introduced two elements depending upon two variables ; they are the *reseau* and the *congruence.* By definition, a point of space in n dimensions describes a *reseau,* if its n rectangular coordinates are solutions of an equation

$$\frac{\partial^2\theta}{\partial u\partial v} = P\frac{\partial\theta}{\partial u} + Q\frac{\partial\theta}{\partial v}.$$

In the customary terminology of differential geometry of ordinary space a reseau is evidently a conjugate system of curves upon the surface locus of these curves. A *congruence* is a doubly infinite system of straight lines which touch two series of curves, as in ordinary space. As thus defined, the lines of the congruence form two families of developable surfaces, a family being given when one of the variables is constant. It is not our purpose to discuss the beautiful theory which Guichard has developed by means of these two elements and set forth in two memoirs, "Les systèmes orthogonaux et les systèmes cycliques" (*Annales de l'Ecole Normale,* 1897, 1898, 1903), but merely to make this reference to it. For it is the application and extension of these methods to the case of triply indeterminate forms which Guichard has developed in the book which is before us for review. This book is more of the nature of a syllabus than a treatise and the matter is so essentially new that it seems advisable to give an outline of its contents rather than attempt a critical review.

In the present theory Guichard makes use of three elements: point systems, line systems and plane systems. The coördinates of a point M in space of n dimensions being given in terms of three parameters u_1, u_2, u_3, M describes a point system if, one of the parameters remaining fixed, the doubly indeterminate system described by M is a reseau. In ordinary space a point system is therefore a triply conjugate system. A line system is formed by the tangents to any one of the curves described by M when only one of the parameters varies. A plane system is described by the plane of two tangents of a point system.

If MT_1 denotes the tangent to the curve of parameter u_1 of a point system M, there are upon MT_1 two determinate points, say A and B, which generate point systems and for which MT_1 is tangent to the curves of parameters u_2 and u_3 respectively. Thus from a given point system we get six other point systems without quadrature, so that we have a transformation of point systems. Moreover, these new point systems are related among themselves in such a way, that if we go, for instance, from M along MT_1 to A, and then along the tangent at A to the curve of parameter u_3, to the point F generating the point system for which AF is tangent at F to the curve of parameter u_1, the line MF is tangent to the curve of parameter u_2 at F. Darboux considered these relations and called them transformations of Laplace (Leçons, IV^e Parte, Chapter XII), as they are generalizations of the transformations of Laplace in the theory of conjugate systems. We have remarked that these transformations form cycles of three. This fact is important as showing that a plane of two tangents to a point system bears the same relation to two other point systems. Consequently, a plane which generates a plane system has three focal points, each of which generates a point system. The lines joining them are the focal lines of the plane system; and the three points, as M, A, B, on a line generating a line system are called its focal points. These definitions and the derivation of the properties of the defined quantities constitute the subject matter of Chapters I and II. The latter contains also the definition of parallel systems. Two point systems are said to be parallel when their corresponding tangents are parallel. Similar definitions are given of the parallelism of line and plane systems. The determination of parallel systems from a given one requires integration. In this connection the following fundamental theorem obtains: if two systems are parallel, the corresponding focal systems are parallel.

With each system, whether point, plane or line, there are associated systems of the other two kinds in the following ways. A line system and a point system are said to be assembled, when the point M lies on the corresponding line Δ and as M describes the curve of parameter u_1 the line Δ envelops a curve of parameter u_1 at the corresponding focal point. A point system and a plane system are assembled when the foci of the plane are situated on the tangents to the corresponding curves of the point system. And a line system and a plane system are assembled when the generating line of the former lies in the generating plane of the latter, the foci of the former lying on the corresponding focal lines of the latter. A fundamental theorem is that when two systems are assembled, their focal elements are assembled. For instance, if a point and line system are assembled, every focal plane of the point is assembled to a focal point of the line. There is also the following law of parallelism: if two systems are parallel, every system assembled with the one is parallel to a system assembled with the other.

In ordinary three dimensional geometry, if we have a triply conjugate system of surfaces, the normals to these surfaces are parallel to the directions of the curves of intersection of other triply conjugate systems, and the determination of the latter requires the integration of a system of partial differential equations of the first order. In Chapter IV, Guichard has generalized this result for space of n dimensions and gathered together his results in a law of orthogonality of elements. This law varies with the order of the space in the following way:

"In spaces of order $3p$, this law makes correspond to a point system, a point system; to a line system, a plane system and inversely. In spaces of order $3p + 1$, it makes correspond to a plane system, a plane system; to a line system, a plane system and inversely. In spaces of order $3p + 2$, it makes correspond to a line system, a line system; to a point system, a plane system and inversely." For spaces of order $3p$ the following theorems obtain:

"If two point systems are orthogonal, every focal line of the one is orthogonal to the focal plane of the same kind of the other. Every line system assembled with the one is orthogonal to a plane system assembled with the other and inversely."

"If a line system and a plane system are orthogonal, every focal point of the line is orthogonal to the focal point of the same kind of the plane; every point system assembled with the line is orthogonal to a point system assembled with the plane and inversely, etc." Similar theorems exist for spaces of order $3p + 1$ and $3p + 2$.

Beginning with Chapter V, the discussion of point systems is concerned chiefly with those for which the tangents to the parametric curves are perpendicular to one another. When these tangents are not isotropic, the system is called a system O; otherwise, a singular rectangular system. The theory of these systems is intimately related to the theory of a determinant Δ whose elements are the coefficients of an orthogonal substitution in space of n dimensions, the elements of the last three rows being the direction cosines of the tangents to the system O and the elements of the other rows being such that their first derivatives are very simple linear functions of the elements of the same columns. The coefficients in these linear functions are called the rotations of the determinant; they must satisfy a system of partial differential equations of the first order. These equations are such that the last three rows of the determinant Δ determine it completely. Consequently to each system O there corresponds a unique determinant, and to a given determinant there corresponds a parallel system of systems O. There are systems O in spaces of any order. The determinant associated with a system O gives a means of finding certain systems assembled with O. When $n > 3$, there are ∞^{n-1} lines, passing through the point M of a system O perpendicular to the three tangents, each of which describes a line system. When $n > 4$ there can be normal line systems generated by isotropic lines. The first kind are called ordinary normal systems; the latter isotropic.

In Chapter VI the notion of the projection of a system from one space into another is introduced. This leads to a generalization of the various systems. Thus, a point M of space of n dimensions describes a system p, O when it can be considered as the projection of a system O of space of $n + p - 1$ dimensions; that is, if $x_1, x_2, \cdots, x_n$ are the coördinates of M, the equations of Laplace which are satisfied by these coordinates admit $p - 1$ other solutions $y_1, \cdots, y_{p-1}$ such that

$$\Sigma dx^2 + \Sigma dy^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2,$$

the functions h_1, h_2, h_3 being different from zero; the quantities $y_1, \cdots, y_{p-1}$ are called the complementary coördinates of the system. A system of lines D is called I, if the sum of the squares of the direction parameters $(X_1, \cdots, X_n)$ of the line D is zero. The isotropic systems of normals to a system O, previously referred to, are of this kind and they are the only ones. When a line system is such that the equations of Laplace satisfied by the quantities X admits $p - 1$ other solutions $(Y_1, \cdots, Y_{p-1})$ such that $\Sigma X^2 + \Sigma Y^2 = 0$, the system is said to be p, I. It is shown that all the ordinary normals to a system O form systems $2I$ and that all other assembled line systems are $3I$. A plane system assembled with a system O is said to be Ω; and it is p, Ω if it can be considered as the projection of a system Ω in space of order $n + p - 1$. Further, it is shown that among the plane systems assembled with a point system $2O$, one series of parallel plane systems are Ω and the others are 2Ω. Thus, there are the following assembled systems for space of three dimensions:

Assembled Systems.

Point.	Plane		Lines.	
O		Ω		3I
$2, O$	1	Ω	1	$3I$
	Others	2Ω	Others	$4I$
$3, O$	2	Ω	2	$3I$
	∞^1	2Ω	∞^1	$4I$
	Others	3Ω	Others	$5I$
p, O	∞^{p-8}	$p-2, \Omega$	∞^{p-3}	$p-2, I$
	∞^{p-2}	$p-1, \Omega$	∞^{p-2}	$p-1, I$
	Others	p, Ω	Others	p, I

There are other assembled systems for three dimensional space, which for lack of space we cannot indicate, and similar tables of assembled systems for spaces of four, five and six dimensions also are given.

In Chapter VII it is shown how the preceding theories enable one to find new triply orthogonal systems. They arise as a result of the search for point, plane and line systems which present two properties; for instance, a point system which is p, O and q, O. In this connection it is found to be advantageous to consider pairs of point systems $M(x_1, \cdots, x_n)$ and $N(y_1 \cdots, y_m)$ which have the property that the Laplace equations are the same for both systems. It results that

$$\Sigma dx^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2,$$

$$\Sigma dy^2 = h_1'^2 du_1^2 + h_2'^2 du_2^2 + h_3'^2 du_3^2,$$

where

$$h_1' = h_1 U_1, \quad h_2' = h_2 U_2, \quad h_3' = h_3 U_3,$$

U_1, U_2, U_3 being functions of u_1, u_2, u_3 respectively. The two systems are said to be associate. Evidently the point whose coördinates are $x_1, x_2, \cdots, y_1, \cdots, y_m$ describes a system O in space of $m + n$ dimensions. Hence a system O associate to a system O in a space of m dimensions is a system O, $(m + 1)\ O$ and inversely. From this it is seen that the systems O, $4O$ are the simplest. By considering systems assembled to a system O, $4O$ in space of three dimensions it is found that the determination of systems $3I$, $4I$; Ω, 2Ω; $2O$, $3O$ are equivalent problems to the determination of systems O, $4O$. Similar and more general results follow from the discussion of systems O, $4O$ in spaces of four, five and six dimensions. And it is found that all problems of finding systems possessing two properties reduces to the search for associate systems O. Interesting particular problems are furnished by considering the cases where certain of the above functions U_1, U_2, U_3 are constants.

As an example of the theory, the determination of systems O, $4O$ in space of three dimensions is treated at length in Chapter VIII and also the analytical expressions for transformations to the assembled systems.

Two point systems are applicable when

$$\Sigma dx^2 = \Sigma dy^2.$$

In the last chapter Guichard considers systems O in three dimensions which are applicable to systems O in six dimensions. It is a study of the determinants of these systems, and from the form of the equations satisfied by the rotations it is found that the problem reduces to the determination of two triply orthogonal systems in space of three dimensions for which the rotations β_{ik} and β'_{ik} of the respective systems satisfy the conditions

$$\beta'_{ik} = \beta_{ki}.$$

The rotations β_{ik} are those defined by Darboux in his Leçons. Of this kind are the systems of Ribaucour for which all the surfaces in one family have the same spherical representation.

We have noticed a few typographical errors which, however, are in most cases errors in reference to sections and not serious. The form of the book is attractive and we should gladly welcome the publication in similar form of the two memoirs on the geometry of two indeterminates, previously mentioned. One who reads these works of Guichard looks forward to the time when he can read them again and try out the ideas which suggest themselves in the reading.

L. P. EISENHART.

SHORTER NOTICES.

Bibliography of Quaternions and Allied Systems of Mathematics. By ALEXANDER MACFARLANE. University Press, Dublin, 1904.

THE International Catalogue of Scientific Literature places the following branches of mathematics under the head of universal algebra: (1) calculus of operations. (2) general theory of complex numbers. (3) quaternions. (4) Ausdehnungslehre and vector analysis. (5) matrices. (6) other special sorts of complex numbers. (7) algebra of logic.

The work under review contains a bibliography of topics 2, 3, 4, and 5. Titles in which vector ideas and methods are applied are also included. Although the subject of bilinear forms is not mentioned in the International Catalogue, Professor Macfarlane very properly includes certain titles in this field which deal with the theory of matrices. It is an extremely difficult matter to prepare a complete bibliography for a field as wide as Professor Macfarlane has chosen, and his approximation will be keenly appreciated, especially by those to whom large libraries are not accessible. The fields of quaternions and vector analysis seem to be carefully covered. In the allied branches of general theory of complex numbers and matrices the bibliography is not so complete, though most of the fundamental memoirs are noted. A partial list of omissions is appended, which contains among others important titles by Weierstrass, Frobenius, Molien and Dedekind.

CARTAN.

1897. "Sur les systèmes de nombres complexes." *Comptes Rendus*, page 1217.

1897. "Sur les systèmes réels de nombres complexes." *Comptes Rendus*, page 1296.

1898. "Les groupes bilinéaires et les systèmes de nombres complexes." *Annales de la Faculté des Sciences de Toulouse*, volume 12, page 81.

DEDEKIND.

1885. "Zur Theorie der aus *n* Haupteinheiten gebildeten complexen Grössen." *Göttinger Nachrichten*, page 141.

1887. "Erläuterungen zur Theorie der sogenannten allgemeinen complexen Grössen." *Göttinger Nachrichten*, page 1.

FROBENIUS.

1896. "Ueber die cogredienten Transformationen der bilinearen Formen." *Sitzungsberichte der Berliner Akademie*, page 7.

HAUSDORFF.

1900. "Zur Theorie der Systeme complexer Zahlen." *Leipziger Berichte*, page 43.

HAWKES.

1902. "Estimate of Peirce's Linear Associative Algebra." *American Journal of Mathematics*, volume 24, page 87.
"On hypercomplex numbers." *Transactions of the American Mathematical Society*, volume 3, page 312.
"Enumeration of non-quaternion number systems." *Mathematische Annalen*, volume 58, page 361.

HILBERT.

1896. "Zur Theorie der aus n Haupteinheiten gebildeten complexen Grössen." *Göttinger Nachrichten*, page 179.

JOLY.

1898. "The associative algebra applicable to hyperspace." *Proceedings of the Irish Academy.*

MOLIEN.

1892. "Ueber Systeme höherer complexer Zahlen." *Mathematische Annalen*, volume 41, page 83.

1893. "Berichtigung zu dem Aufsatze: Ueber Systeme höherer complexer Zahlen." *Mathematische Annalen*, volume 42, page 308.

PIERCE.

1903. "Classification and properties of dual conical congruences." Dissertation, Zürich.

STUDY.

1890. "Ueber Systeme complexer Zahlen und ihre Anwendung in der Theorie der Transformationsgruppen." *Monatshefte für Mathematik*, page 283.

1898. "Beweis eines von Herrn Dedekind angegebenen Satzes." *Göttinger Nachrichten*, page 1.

TABER.

1892. "On a theorem of Sylvester's relating to non-degenerate matrices." *Proceedings of the American Academy*, page 46.

"Note on the representation of orthogonal matrices." *Proceedings of the American Academy*, page 163.

1893. "On real orthogonal substitutions." *Proceedings of the American Academy*, page 212.

1894. "On the automorphic linear transformation of bilinear forms." *Proceedings of the American Academy*, page 178.

"On orthogonal substitutions." *Bulletin of the New York Mathematical Society*, volume 3, page 251.

1897. "Notes on the theory of bilinear forms." *Bulletin of the American Mathematical Society*, volume 3, page 156.

WEIERSTRASS.

1884. "Zur Theorie der aus n Haupteinheiten gebildeten complexen Zahlen." *Göttinger Nachrichten*, page 395.

H. E. HAWKES.

An Elementary Text-book on the Differential and Integral Calculus. By WILLIAM H. ECHOLS. Professor of Mathematics in the University of Virginia. New York, Henry Holt and Company, 1902.

IN this book Professor Echols has made a commendable and, on the whole, a successful attempt to establish the principles of the differential and integral calculus with proper regard to modern ideas on the subject, pointing out carefully the limits under which the processes can be applied and giving examples of exceptional cases, such as Weierstrass's example of a continuous function which has no determinate derivative, and Pringsheim's example of a function for which Maclaurin's series is absolutely convergent and yet does not represent the function. One would like to see also, however, an example in which $\partial^2z/\partial x\partial y$ and $\partial^2z/\partial y\partial x$ are different — a case of peculiar interest in some of the applications.

The book is far more comprehensive than the usual textbooks, and should therefore be useful for consultation by students in the later years of their course. The author states that it is intended to cover about one year's work, but it seems doubtful if it could be completed in a much longer period in many colleges. However, there should be no trouble in making a judicious selection from the wealth of material and an ambitious student would be glad to possess a book which would give him so much opportunity for work outside of the classroom.

Special features of the book are the complete separation of functions of one variable from those of more than one, the liberal use of the theorems of mean value, the definition of an integral primarily as the limit of a sum, a definition to which it may be hoped all of our text-books will ultimately return, a careful introduction on variables and functions and a number of notes at the end including the examples referred to.

Where there is so much to commend it may seem invidious to criticize, but it seems to the reviewer that the definition of transfinite numbers and of infinitesimals is likely to be misunderstood by the beginner; that it would be well to print the definition of irrational numbers in larger type and let it precede the employment of such numbers; that something would be gained in the treatment of trigonometric integrals by the introduction into the text of more of the theory of the complex variable, including at least De Moivre's theorem. It is also somewhat surprising not to find a definition of curvature—the radius of curvature being defined merely as the radius of the osculating circle.

The reviewer is disposed to agree with the author in his relegation of the well-known and too much worshiped reduction formulas for the integral

$$\int x^{\alpha}(a + bx^{n})^{\gamma}dx$$

to a note at the end of the book.

M. W. HASKELL.

American Men of Science. A Biographical Directory. Edited by J. McKEEN CATTELL. New York, The Science Press, 1906. 4to. 7 + 364 pp.

THE chief object of this directory is to make American "men of science acquainted with one another and with one another's work." In the field which it aims to cover it is much more comprehensive than Minerva or Who's Who in America, as it includes a tolerably complete list of sketches of those living in North America who have published any research in the natural and exact sciences, and in addition some entries of those who are held to have advanced science by teaching, by administrative work, or by the publication of text-books.

In the case of about one thousand out of the four thousand biographical notes a star is placed before the field of research. This indicates that the subject of the sketch is to be regarded as one of the thousand leading American men of science. About 80 of the 360 entries of mathematicians are distinguished in this manner. Those who are inclined to estimate the scientific standing of a man by the post which he fills will be surprised at the large number of names of younger men whose work is starred, while many who hold prominent positions are not classed among the leading scientists.

In view of the great difficulties in forming a correct estimate of the merits of the work of our contemporaries, the present undertaking appears a bold one, but the importance of accurate knowledge on this point seems to justify such ventures. All men of science and especially those who require the services of such men must feel keenly the difficulties encountered in the effort of judging the relative merits of the work of those who may be under consideration; and as such judgments are sometimes imperatively necessary, any reliable aid is a great desideratum. Moreover, there are few other things which tend to contribute so much toward substantial progress as a general feeling that the opinion of experts in regard to individual achievements will speedily become common property. The present volume appears to be a decided step towards cultivating such a feeling.

As far as possible each entry contains information on the following ten points: Name and mail address, department of investigation, place and date of birth, education and degrees, positions with dates, temporary and minor positions, honorary degrees and other scientific honors, membership in learned societies, chief subjects of research, and whether the subject of the sketch is classed with "the thousand students of the natural and exact sciences in the United States whose work is supposed to be the most important."

G. A. MILLER.

NOTES.

THE tenth regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY will be held at the University of California on Saturday, September 29.

THE July number (volume 7, number 3) of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "On the boundary value problems of linear ordinary differential equations of second order," by M. MASON; "The resolution of any collineation into perspective reflections," by M. W. HASKELL; "Linear algebras in which division is always uniquely possible," by L. E. DICKSON; "Correspondences and the theory of groups," by J. E. WRIGHT; "The trajectories of dynamics," by E. KASNER; "On the automorphic functions of the group $(0, 3; l_1, l_2, l_3)$," by R. MORRIS; "Improper multiple integrals," by R. G. D. RICHARDSON.

THE April number (volume 28, number 2) of the *American Journal of Mathematics* contains: "The motion of a solid in an infinite liquid," by A. G. GREENHILL; "The theory of implication," by BERTRAND RUSSELL.

The July number (volume 28, number 3) contains: "The geometry of differential elements of the second order with respect to the group of all point transformations," by EDWARD KASNER; "Gyroscopes and cyclones," by F. J. B. CORDEIRO; "On the primitive groups of class ten," by W. A. MANNING; "On certain unicursal twisted curves," by VIRGIL SNYDER; "Functions of three real independent variables," by H. L. COAR.

THE concluding (July) number of volume 7 of the *Annals of Mathematics* contains: "A proof of the theorem concerning artificial singularities," by D. R. CURTISS; "Another proof of the theorem concerning artificial singularities," by M. BÔCHER; "Curves of minimum moment of inertia with respect to a point," by M. MASON; "Triangles and quadrilaterals inscribed to a cubic and circumscribed to a conic," by H. S. WHITE; "On a function which occurs in the law of the mean," by E. R. HEDRICK.

THE annual list of American doctorates published in *Science* presents for the academic year 1905–1906 325 names, of which 139 are credited to the sciences. The following 9 successful candidates offered mathematics as major subject (the titles of the theses are appended): E. C. COLPITTS, Cornell, "On the twisted quintic curves"; B. F. FINKEL, Pennsylvania, "Determination of all groups of order 2 which contain cyclic self-conjugate subgroups of order 2 and whose generating operators correspond to the partitions"; C. C. GROVE, Johns Hopkins, "I. The syzygetic pencil of cubics and a new geometrical development of its Hesse group G_{216}. II. On the complete Pappus hexagon"; H. B. LEONARD, Colorado, "On the factoring of composite algebras"; J. F. MESSICK, Johns Hopkins, "Cubic curves in reciprocal triangular situation"; R. G. D. RICHARDSON, Yale, "Improper multiple integrals"; W. H. ROEVER, Harvard, "Brilliant points"; C. H. SISAM, Cornell, "Ruled surfaces of order seven having a rectilinear directrix"; G. E. WAHLIN, Yale, "The relation between the binary quadratic forms and the quadratic numerical bodies."

The number of American doctorates in mathematics for each year from 1898 to 1906 is 11, 13, 11, 18, 8, 7, 14, 21, 9.

THE first supplementary volume of the *Jahresberichte der deutschen Mathematiker-Vereinigung*, containing "Ueber die Entwicklung der Elementar-Geometrie im XIX Jahrhundert," by M. SIMON, has just been published by Teubner in Leipzig.

IN connection with the recent division of the *Proceedings of the Royal Society of London* into Series A, papers of a mathematical and physical character, and Series B, papers of a biological character, attention is called to the arrangement by which, on the advance payment of the subscription price of 15 shillings per volume, subscribers may receive the separate papers as soon as published.

SYRACUSE UNIVERSITY — The following advanced courses in mathematics are offered during the first semester of the current academic year: — By Professor W. H. METZLER: Analytic geometry, plane and solid, three hours; Newtonian potential function, three hours; Elliptic functions, three hours; Weierstrass's theory of functions, three hours. — By Professor E. D. ROE: Determinants, one hour; Theory of equations, two hours; Advanced calculus, three hours; Analytic mechanics, three hours; Symmetric functions; three hours; Invariants, three hours; Substitutions, three hours. — By Professor W. G. BULLARD: Projective geometry, three hours; Curves and surfaces, three hours; Differential geometry, three hours.

THE various foreign universities offer courses in mathematics during the winter semester of 1906–1907 as follows:

UNIVERSITY OF GÖTTINGEN. — By Professor F. KLEIN: Elliptic functions, four hours; Seminar (with Professors Hilbert and Minkowski and Dr. Herglotz), two hours. — By Professor D. HILBERT: Mechanics of continua, four hours; Differential and integral calculus, four hours. — By Professor H. MINKOWSKI: Encyclopedia of elementary mathematics, four hours; Invariants, four hours. — By Professor C. RUNGE: Descriptive geometry, four hours; with exercises, two hours. — By Professor M. BRENDEL: Mathematics of insurance, two hours. — By Professor E. ZERMELO: Elements of analytic mechanics, four hours; Mathematical treatment of logic, one hour. — By Dr. M. ABRAHAM: Partial differential equations of mathematical physics, four hours; Seminar, two hours. — By Dr. C. CARATHÉODORY: Minimum principles of mechanics and physics, one hour. — By Dr. G. HERGLOTZ: Introduction to analytic geometry of space, four hours.

UNIVERSITY OF JENA. — By Professor J. THOMAE: Definite integrals, four hours; Differential equations, four hours; Seminar, two hours. — By Professor F. HAUSSNER: Mathematical approximations, two hours; Integral calculus with exercises, five hours; Analytic geometry of space, four hours; Seminar, one hour. — By Professor G. FREGE: Theory of functions according to Riemann, four hours. — By Professor R. RAU: Mechanics, four hours; Descriptive geometry with exercises, seven hours.

UNIVERSITY OF LEIPZIG. — By Professor C. NEUMANN: Analytic mechanics, four hours; Seminar, one hour. — By Professor A. MAYER: Calculus of variations, four hours. — By Professor O. HÖLDER: Differential and integral calculus, five hours; Foundations of arithmetic and numbers, two hours; Seminar, one hour. — By Professor K. ROHN: Applications of the calculus to surfaces and twisted curves, four hours; Invariants, two hours. — By Professor F. HAUSDORFF: Theory of numbers, four hours. — By Professor H. LIEBMANN: Analytic geometry of space, two hours; with exercises, one hour.

UNIVERSITY OF MUNICH. — By Professor F. LINDEMANN: Theory of functions of a complex variable, four hours; Applications of the calculus to curves and surfaces, four hours; Groups of transformations, two hours; Seminar, one and a half hours. — By Professor A. VOSS: Algebra, four hours; Theory of algebraic curves, four hours; Seminar, two hours. — By Professor A. PRINGSHEIM: Differential calculus, five hours; Theory of numbers, four hours. — By Professor K. DOEHLEMANN: Descriptive geometry, I, five hours; with exercises, three hours; Line geometry, four hours. — By Professor E. v. WEBER: Plane analytic geometry, four hours; with exercises, two hours; Integral calculus, with exercises, four hours. — By Dr. H. BRUNN: Theory of aggregates, four hours. — By Dr. F. HARTOGS: Selected chapters of the theory of functions, two hours. — By Dr. O. PERRON: — not yet announced.

OXFORD UNIVERSITY (Michaelmas Term). — By Professor W. ESSON: Analytic geometry of plane curves, two hours; Synthetic geometry of plane curves, one hour. — By Professor E. B. ELLIOTT: Sequences and series, two hours; Elementary theory of numbers, one hour. — By Professor A. E. H. LOVE: Hydrodynamics, two hours; Problems in applied mathematics, one hour. — By Professor H. H. TURNER: Elementary mathematical astronomy, two hours. — By Mr. C. E. HASELFOOT: Theory of equations, one hour. — By Mr. C. LEUDESDORF: Projective geometry, three hours. — By Mr. A. E. JOLLIFFE: Analytic geometry, two hours. — By Mr. J. W. RUSSELL: Differential calculus, two hours. — By Mr. R. F. MCNEILE: Curve tracing, one hour. — By Mr. A. L. PEDDER: Problems in pure mathematics, one hour. — By Mr. C. H. SAMPSON: Higher solid geometry, two hours. — By Mr. J. E. CAMPBELL: Differential equations, two hours. — By Mr. C. H. THOMPSON: Integral calculus, two hours. — By Mr. E. H. HAYES: Analytical statics, three hours. — By Mr. A. L. DIXON: Hydrostatics, one hour. — By Mr. H. T. GERRANS: Tridimensional rigid dynamics, two hours. — By Mr. P. J. KIRKBY: Attractions and electrostatics, two hours.

THE royal academy of sciences at Berlin has awarded Professor F. MERTENS, of the University of Vienna, a prize of 5000 marks for his researches in the theory of cyclic equations.

Professor E. Müller, of the technical school at Vienna, has been elected a corresponding member of the royal academy of sciences at Vienna.

Dr. F. Jung, docent of mathematics at the German technical school at Prague, has been transferred to a similar position at the technical school at Vienna.

Professor G. Scheffers, of Darmstadt, has been appointed professor of mathematics at the technical school of Charlottenburg.

Dr. L. Hanni has been appointed docent in mathematics at the University of Vienna and Dr. O. Perron at the University of Munich.

Mr. B. A. Herman and Mr. H. W. Richmond have been appointed university lecturers in mathematics at Cambridge University.

At the University of California Professor M. W. Haskell has been promoted to a full professorship of mathematics, and Professor E. J. Wilczynski to an associate professorship of mathematics.

At Stanford University Professor H. F. Blichfeldt has been promoted to an associate professorship of mathematics, and Dr. W. O. Mendenhall has been appointed instructor in applied mathematics.

Professor H. P. Manning, of Brown University, has been promoted to an associate professorship of mathematics.

At Lehigh University Professors P. A. Lambert and A. E. Meake have been promoted to full professorships of mathematics.

At the University of Rochester, Dr. A. S. Gale has been promoted to the newly designated Fayerweather professorship of mathematics.

Professor T. E. McKinney, of Marietta College, has been appointed acting professor of mathematics at Wesleyan University.

Mr. P. P. Boyd has been appointed professor of mathematics at Hanover College, Indiana.

MR. W. J. NEWLIN has been appointed associate professor of mathematics and philosophy at Amherst College.

REV. R. D. CARMICHAEL has been appointed professor of mathematics at the Presbyterian College for Men at Anniston, Alabama.

DR. J. T. ROOD has been appointed professor of mathematics at Ursinus College, Collegeville, Pennsylvania.

DR. C. C. GROVE has been appointed assistant professor of mathematics at Hamilton College, Clinton, New York.

DR. H. L. COAR, of the University of Illinois, has been appointed associate professor of mathematics at Marietta College, Marietta, Ohio.

DR. D. C. GILLESPIE has been appointed instructor in mathematics at Cornell University.

DR. G. E. WAHLIN has been appointed instructor in mathematics at the University of Iowa.

MR. J. M. THORNTON has been appointed professor of mathematics at the University of West Virginia.

DR. E. L. DODD has been appointed instructor in mathematics at the University of Illinois.

MR. E. R. SLEIGHT has been appointed professor of mathematics at Carthage College, Carthage, Illinois.

DR. W. R. LONGLEY has been appointed instructor in mathematics and astronomy at Yale University.

NEW PUBLICATIONS.

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ADAM (C.). See DESCARTES (R.).

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BOCHOW (K.). Die Funktionen rationaler Winkel; insbesondere über die Berechnung der Winkelfunktionen ohne Benutzung der trigonometrischen Reihen und der Zahl π. (Progr.) Magdeburg, 1905.

BONOLA (R.). La geometria non-euclidea: esposizione storico-critica de suo sviluppo. Bologna, Zanichelli, 1906. 8vo. 6 + 213 pp. L. 5.00

BRÓHL (J.). Doppelt unendliche Kurvenscharen, gebildet von Schraubenlinien auf den Flächen der Rotationsflächenschar $x^2 + y^2 = u^2e^{2x}$. Munster, 1905. 8vo. 53 pp. M. 2.00

BRÜGMANN (W.). Ueber eine reell irreducible Gruppe von Berührungstransformationen. (Progr.) Hamburg, 1906. 4to. 35 pp.

BRUNN (H.). Beziehungen des du Bois-Reymondschen Mittelwertsatzes zur Ovaltheorie. Eine mathematische Studie. Berlin, Reimer, 1905. 8vo. 10 + 135 pp.

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FUETER (R.). Die Theorie der Zahlstrahlen (Hab.) Marburg, 1905. 4to. 41 pp.

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GASSER (A.). Ueber die Nullstellen der Besselschen Funktionen. (Diss.) Bern, Wyss, 1904. 8vo 48 pp., 2 plates.

GROSSMANN (M.). Analytische Geometrie. Leitfaden für den Unterricht an höheren Lehranstalten. Basel, Helbing, 1906. 8vo. 104 pp. Boards. M. 2.00

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HALSTED (G. B.). Synthetic projective geometry. [2d edition.] New York, Wiley, 1906. 8vo. 37 pp. Cloth. (Mathematical monographs, edited by M. Merriman and R. S. Woodward No. 2.) $1.00

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HOCHHEIM (A.). Aufgaben aus der analytischen Geometrie der Ebene. Heft 2: Die Kegelschnitte. Abteilung 1, A: Aufgaben. 3te, vermehrte Auflage, bearbeitet von O. Jahn und F. Hochheim. Leipzig, Teubner, 1906. 8vo. 90 pp. M. 1.80

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Lucas de Peslouan (C.). N.-H. Abel Sa vie et son œuvre. Paris, Gauthier-Villars, 1906. 12mo. 13 + 170 pp., portrait. Fr. 5.00

Macfarlane (A.). Vector analysis and quaternions. [2d edition.] New York, Wiley, 1906. 8vo. 33 pp. Cloth. (Mathematical monographs, edited by M. Merriman and R. S. Woodward, No. 8.) $1.00

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Tannery (P.). See Descartes (R.).

Thornburg (C. L.). Calculus notes. Lancaster, Pa., New Era Printing Co., 1906. 12mo. 33 pp. Cloth. $0.50

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WELD (L. G.). Determinants. [2d edition.] New York, Wiley, 1906. 8vo. 37 pp. Cloth. (Mathematical monographs, edited by M. Merriman and R. S. Woodward, No. 3.) $1.00

WHITEHEAD (A. N.). Mathematical concepts of the material world. London, Dulau, 1906. 4ts. 60 pp. 2s. 6d

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BLOCK (C.). Lehr- und Uebungsbuch fur den planimetrischen Unterricht an hoheren Schulen. Teil 2 : Untertertia. Leipzig, Teubner, 1906. 8vo. M. 0.80

BÖKELER (A.). Stereometrische Aufgaben aus den Reifeprufungen der Gymnasial-Abiturienten. Auflosungen. Ravensburg, Alber, 1906. 8vo. 4 + 80 pp. M. 2 00

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BORTOLOTTI (E.). Nozioni di geometria per le scuole tecniche : guida ad esami di geometria. Roma, Voghera, 1906. 8vo. 280 pp.

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CAREY (F. S.) Elementary solid geometry. Enlarged edition. London, Arnld, 1906. 12mo. 124 pp. Cloth. 2s. 6d.

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DELILLE (A.). See LEGENDRE.

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GAJDECZKA (J.). Uebungsbuch zur Arithmetik und Algebra fur die oberen Klassen der Mittelschulen. 7te Auflage. Wien, Tempsky, 1906. 8vo. 230 pp. Cloth. M. 3.00

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THE THIRTEENTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE thirteenth summer meeting and the fifth colloquium of the Society were held at Yale University during the week September 3–8, 1906. The following forty-six members were in attendance:

Mr. G. D. Birkhoff, Professor G. A. Bliss, Professor Joseph Bowden, Professor E. W. Brown, Dr. A. B. Chace, Dr. J. E. Clarke, Dr. A. Cohen, Professor F. N. Cole, Professor L. L. Conant, Professor D. R. Curtiss, Mr. H. N. Davis, Professor L. P. Eisenhart, Professor W. B. Fite, Professor A. S. Gale, Professor C. N. Haskins, Dr. L. I. Hewes, Professor Edward Kasner, Professor O. D. Kellogg, Dr. W. R. Longley, Mr. E. B. Lytle, Professor T. E. McKinney, Professor James McMahon, Professor H. P. Manning, Professor Max Mason, Mr. E. A. Miller, Professor E. H. Moore, Professor W. F. Osgood, Professor James Pierpont, Dr. R. G. D. Richardson, Miss S. F. Richardson, Professor H. L. Rietz, Miss I. M. Schottenfels, Mr. A. R. Schweitzer, Dr. C. H. Sisam, Professor C. S. Slichter, Dr. Clara E. Smith, Professor P. F. Smith, Professor Virgil Snyder, Professor H. F. Stecker, Dr. R. P. Stephens, Professor W. E. Story, Professor E. B. Van Vleck, Professor A. G. Webster, Professor H. S. White, Professor T. W. D. Worthen, Professor J. W. Young.

The colloquium, of which a separate report appears in the BULLETIN, opened on Wednesday morning. The summer meeting proper extended through two sessions on Monday and a morning session on Tuesday. On Tuesday afternoon the visitors were conducted through the grounds and buildings of the University. The generous hospitality extended to the Society throughout the meeting by the University and its officers will long be remembered with grateful appreciation.

At the opening session on Monday morning Professor E. H. Moore presided. President W. F. Osgood occupied the chair during the remaining sessions. The Council announced the election of the following persons to membership in the Society: Professor William Beebe, Yale University; Mr. J. B. Clarke, Polytechnic High School, San Francisco; Dr. E. C. Colpitts, Cornell University; Brother Constantius, Christian Brothers College, St. Louis; Professor G. W. Droke, University of Arkansas; Mr. R. M. Ginnings, State Normal School, Kirksville, Mo.. Professor Harriet E. Glazier, Western College for Women; Professor C. O. Gunther, Stevens Institute of Technology; Mr. W. A. Hurwitz, University of Missouri; Dr. G. O. James, Washington University; Mr. B. F. Johnson, State Normal School, Cape Girardeau, Mo.; Mr. E. B. Morrow, Princeton University; Mr. G. B. Obear, Brown University; Dr. F. M. Pedersen, College of the City of New York; Professor G. A. Rose, Hardin College; Mr. R. L. Short, Chicago, Ill.; Miss Betty Trier, Mount Holyoke College; President J. W. Withers, Teachers College, St. Louis. Thirteen applications for membership in the Society were received.

A committee consisting of Professors Bôcher, E. B. Van Vleck, and Townsend was appointed by the President to prepare and report to the Council at the October meeting a list of nominations for officers and other members of the Council to be elected at the annual meeting in December. Steps were also taken toward amending the Constitution to include the Editorial Committee of the *Transactions* in the membership of the Council.

The following papers were read at the summer meeting:

(1) Mr. A. R. SCHWEITZER: "Systems of axioms for projective geometry."

(2) Mr. A. R. SCHWEITZER: "Concerning abstract geometric relations" (preliminary report).

(3) Professor O. D. KELLOGG: "The behavior on the boundary of harmonic functions of a region."

(4) Mr. F. R. SHARPE: "The motion of a viscous gas."

(5) Professor R. D. CARMICHAEL: "Multiply perfect numbers of three different primes."

(6) Professor LUDWIG STICKELBERGER: "Zur Theorie der vollständig reduciblen Gruppen, die zu einer Gruppe linearer homogener Substitutionen gehören."

(7) Professor W. B. FITE: "Irreducible linear homogeneous groups whose orders are powers of a prime."

(8) Dr. ARTHUR RANUM: "The group of classes of congruent matrices and its application to the group of isomorphisms of any abelian group."

(9) Dr. R. G. D. RICHARDSON: "On the reduction of multiple integrals (second paper)."

(10) Mr. G. D. BIRKHOFF: "On a certain class of sets of normed orthogonal functions."

(11) Dr. W. B. CARVER: "Associated configurations of the Cayley-Veronese class."

(12) Professor L. E. DICKSON: "On commutative linear algebras in which division is always uniquely possible."

(13) Professor L. E. DICKSON: "Uniform definitions of the abstract forms of the various known systems of linear groups."

(14) Professor L. E. DICKSON: "Criteria for the irreducibility of functions in a finite field."

(15) Professor L. E. DICKSON: "On the theory of equations in a modular field."

(16) Professor JAMES MCMAHON: "The differential geometry of the general vector field" (preliminary report).

(17) Dr. W. A. MANNING: "A note on transitive groups."

(18) Dr. C. H. SISAM: "On systems of conics lying on surfaces of the third, fourth, and fifth orders."

(19) Professor VIRGIL SNYDER: "Plane quintic curves which possess a group of linear transformations."

(20) Professor MAX MASON: "The expansion of an arbitrary function in terms of normal functions."

(21) Professor MAX MASON: "The boundary value problems of differential equations of hyperbolic type."

(22) Professor EDWARD KASNER: "An inverse problem of dynamics."

(23) Professor EDWARD KASNER: "The geometry of dynamical trajectories."

(24) Professor J. W. YOUNG: "General theory of approximation by functions with a given number of parameters."

(25) Professor J. I. HUTCHINSON: "On loci the coordinates of whose points are abelian functions of three parameters."

(26) Professor L. P. EISENHART: "Applicable surfaces with asymptotic lines of one surface corresponding to a conjugate system of another."

(27) Dr. H. B. LEONARD: "On the factoring of composite hypercomplex number systems."

(28) Professor FRANK MORLEY: "Reflexive geometry."

(29) Professor G. A. MILLER: Generalization of the groups of genus zero."

(30) Professor E. B. WILSON: "On divergence and curl."

(31) Professor E. B. WILSON: "Oblique reflections and unimodular strains."

(32) Professor E. B. WILSON: Double products and strains in n dimensions."

(33) Professor F. R. MOULTON: "A class of three dimensional periodic orbits in the problem of three bodies, with applications to the lunar theory."

(34) Professor OSKAR BOLZA: "Weierstrass's theorem and Kneser's theorem on transversals for the most general case of an extremum of a simple definite integral."

Professor Stickelberger's paper was communicated to the Society through Professor E. H. Moore. In the absence of the authors, Mr. Sharpe's paper was presented by Professor McMahon, Dr. Ranum's by Professor Moore, Dr. Carver's by Professor Snyder, Professor Hutchinson's by Professor Fite, and the papers of Professor Carmichael, Professor Stickelberger, Professor Dickson, Dr. Manning, Dr. Leonard, Professor Morley, Professor Miller, Professor Wilson, Professor Moulton and Professor Bolza were presented by title.

The third and fourth papers of Professor Dickson and the paper of Dr. Manning appeared in full in the October BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Mr. Schweitzer constructed systems of axioms for projective geometry of n dimensions ($n = 1, 2, 3, \cdots$), using two undefined symbols, viz., the element point and relation K_n between two ordered $(n + 2)$-ads of points. Under the respective systems the relation K_n is symmetric and transitive, *i. e.*, $(A_1A_2\cdots A_{n+2})K_n(B_1B_2\cdots B_{n+2})$ implies $(B_1B_2\cdots B_{n+2})K_n(A_1A_2\cdots A_{n+2})$ and the statements $(A_1A_2\cdots A_{n+2})K_n(B_1B_2\cdots B_{n+2})$, $(B_1B_2\cdots B_{n+2})K_n(C_1C_2\cdots C_{n+2})$ imply $(A_1A_2\cdots A_{n+2})K_n(C_1C_2\cdots C_{n+2})$. The existence of an ordered $(n + 2)$-ad implies no geometric axiom except the existence of points. Systems III, IV, $\cdots$ are sufficient for 3, 4, $\cdots$ dimensional projective geometry respectively. Systems, I, II, $\cdots$ are readily extended to higher dimensions. In each of the preceding systems, collinearity, separation, etc., are defined in terms of the corresponding relation K_n.

The above sets of axioms may be considered the projective analogues of the author's systems of descriptive axioms* for

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* Cf. abstract No. 6, BULLETIN, June, 1906, p. 438; for n-ad read $(n+1)$-ad. It should also there be stated that under the axioms the K_n is symmetric and transitive in the sense indicated above.

n dimensions ($n = 1, 2, 3, \cdots$) involving the two undefined elements point and the symmetric, transitive relation K_n between two ordered $(n + 1)$-ads of points. A result of both investigations is that it is possible to construct axioms for n dimensions, ($n = 1, 2, 3, \cdots$), which may be either descriptive or projective by means of two undefined symbols point and a transitive, symmetric relation; and in n-dimensional space this relation may be either linear, or planar, $\cdots$, or n-dimensional.

2. In his second paper Mr. Schweitzer investigates the kinds of undefined relations between points which may be used in the construction of geometric axioms, for example: 1) $\alpha R_n(\beta_1\beta_2 \cdots \beta_n)$, ($n = 1, 2, 3, \cdots$), where the n-ad is ordered; 2) $[\alpha_1 \cdots \alpha_n] B_n [\beta_1\beta_2]$, ($n = 1, 2, 3, \cdots$), where the dyad and n-ad are not ordered; 3) $(\alpha_1\alpha_2) S_{n+2}(\beta_1\beta_2 \cdots \beta_{n+1})$($n=1, 2, 3, \cdots$), where the dyad and $(n + 1)$-ad are both ordered. In particular, the construction of descriptive axioms in terms of the relation R_n under 1) leads to essential simplifications of a treatment of geometric order given by Vahlen in his "Abstrakte Geometrie." This is shown in a note which will appear in the BULLETIN.

3. The object of Professor Kellogg's paper is the study of the behavior on the boundary of the derivatives of harmonic functions of a region, and allied questions. Some preliminary results for the circle are here given. The formulas of Hilbert,

$$f(s) = \int_0^1 g(t) \cot \pi(s - t)dt + \int_0^1 f(t)dt,$$

$$g(s) = -\int_0^1 f(t) \cot \pi(s - t)dt + \int_0^1 g(t)dt,$$

giving the boundary values of a harmonic function of the unit circle in terms of those of the conjugate function, still hold if we only postulate piecewise regularity of $f(s)$ and $g(s)$. By regularity we mean that a condition holds of the form $|f(s + h) - f(s)| < Ch^\alpha$, where C and α are positive constants, C as large and α as small as we please. A study of these formulas leads to the results: It is not sufficient for the continuity of the boundary values of a harmonic function that the boundary values of the conjugate function be continuous. It is sufficient that they be regular. In order that any first derivative of a harmonic function be continuous up to and upon the boundary, it is not sufficient that the tangential derivatives on the boundary be continuous. It is sufficient that they be regular. (This last result is proved for the circle in a slightly more general form by Dini, *Acta Mathematica*, volume 25, page 221 and following.) The necessary and sufficient conditions can be brought even nearer together, but it seems scarcely worth while. The extension of these results to more general regions will be the subject of a later report.

Another remark concerns the distinction between the conformality of the mapping of a function of a complex variable and the finiteness and non-vanishing of the derivative. That these are not the same is at once seen by an examination of the mapping given by $\zeta = z \log z$ in the neighborhood of the origin. The paper will be offered to the *Transactions* for publication.

4. In discussing the motion of a gas it is usually assumed that every element neither gains nor loses heat, and consequently the pressure varies as some power of the density so long as we consider elements on the same stream line. In Mr. Sharpe's paper the adiabatic assumption is replaced by a new equation obtained by considering the total flow of energy, molar and molecular, which includes the effects of viscosity and conduction of heat. It is shown that this equation agrees with a similar one deduced from the kinetic theory of gases, and it is applied to the following problems: 1) heated air rising up a chimney, 2) gas flowing between two smooth parallel planes, 3) gas flowing between two rough parallel planes, 4) gas surrounding a rotating circular cylinder and inside a fixed concentric cylinder.

5. In this paper Professor Carmichael completes the investigation of the existence of multiply perfect numbers of (only) three different prime factors, showing that the only such numbers are $2^3 \cdot 3 \cdot 5$ and $2^5 \cdot 3 \cdot 7$, each of multiplicity 3. In a footnote he also points out that there exists no multiply perfect number which is the power of a prime, and that all perfect numbers of (only) two different primes are of the form $2^{n-1}(2^n - 1)$, where $2^n - 1$ is prime. The paper will be published in the *Annals of Mathematics*.

6. Professor Stickelberger gives a very simple proof of a lemma forming the basis of the article by Professor Loewy,

"Über die vollständig reduciblen Gruppen, die zu einer Gruppe linearer homogener Substitutionen gehören" (*Transactions*, volume 6 (1905), pages 504–533). Moreover, there results an extension of Loewy's theorem. The paper will appear in the *Transactions*.

7. Certain irreducible linear homogeneous groups of different degrees are simply isomorphic; whereas, on the other hand, all the irreducible groups with which some groups are simply isomorphic are of the same degree. The question thus suggested as to the connection between the degree of an irreducible group and its abstract group properties forms the subject of Professor Fite's paper. The discussion is limited to groups whose orders are powers of a prime.

8. With respect to the n^2 moduli $p^{a_{ij}}(i, j=1, \cdots n;\ p=\text{a prime})$ two matrices (l_{ij}) and (m_{ij}) are said to be congruent, if $l_{ij} \equiv m_{ij}$, mod $p^{a_{ij}}$. Dr. Ranum proves that the composition of classes of congruent matrices is possible if, and only if, the moduli satisfy the condition $a_{ij} + a_{jk} \geqq a_{ik}$ $(i,\ j,\ k = 1, \cdots, n)$ and the elements l_{ij} are divisible by $p^{\alpha_{ij}}$, where α_{ij} is the greatest of the $2n + 1$ integers $0,\ a_{ik} - a_{jk},\ a_{kj} - a_{ki}$ $(k = 1, \cdots, n)$. If these conditions are satisfied, the totality of classes of matrices whose determinants are prime to p form a group, of which all other groups satisfying the same condition are subgroups. The order, invariant matrices, and composition series of this group, and its largest invariant subgroup of order a power of p, are found. The application of a special group of this kind to the group of isomorphisms of any abelian group is an amplification of a paper read before the San Francisco Section at its February meeting, entitled "A new kind of congruence group, etc." (abstract in BULLETIN, May, 1906, page 375).

9. In a former paper (April, 1906), Dr. Richardson showed that in all cases the existence of the double integral and the iterated integral is a sufficient condition for their equality. The present paper discusses the corresponding problem which arises when either the double integral or the iterated integral exists. In the *Journal de Mathématiques*, series 5, volume 5 (1899), de la Vallée-Poussin derived the relation

$$1)\quad \overline{\int_T} f dT \geqq \underline{\int_y} dy \overline{\int_x} mf dx \geqq \underline{\int_y} dy \underline{\int_x} mf dx \geqq \underline{\int_T} f dT \qquad (f \geqq 0),$$

where T is a portion of the plane bounded by a regular curve and the minimum function mf is defined at each point as the minimum of f in an infinitesimal area about that point. The introduction of the minimum function is shown to be unnecessary and relation 1) is true if f is substituted for mf. If the double integral exists, this becomes

$$\int_T f dT = \int_{\underline{y}} dy \overline{\int_x} f dx = \int_{\underline{y}} dy \int_{\underline{x}} f dx.$$

Other relations are derived under the hypothesis that the iterated integral alone exists. For the generalized integral introduced by Professor Pierpont, the theorems are valid; and if the fields of integration are $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$, of $m + n$, m, and n dimensions respectively, the discussion applies. Finally, the theorems are extended to the case where the field of integration is unlimited.

10. Mr. Birkhoff's paper deals with certain infinite sets of normed orthogonal functions, $\phi_1(x)$, $\phi_2(x)$, $\cdots$, *i. e.*, sets of functions ϕ such that

$$\int_0^1 \phi_i(t)\phi_j(t)dt = \begin{matrix} 0, & (i \neq j) \\ 1, & (i = j) \end{matrix};$$

such a set is defined as the set of normed solutions of an equation

$$\phi(x) = \lambda \int_0^1 k(x, t)\phi(t)dt,$$

where λ is a parameter and where 1) $k(x, t)$ is defined for $0 \leqq x, t \leqq 1$ and is real, continuous, and symmetric in x and t; 2) $k(x, t)$ has continuous partial derivatives in x, t of first and second orders, except for $x = t$, where there is the discontinuity of a finite jump, at least in the first partial derivatives.

The principal theorems deal with the possibility of representing a prescribed function $f(x)$ on the interval $(0, 1)$ as a series in the functions ϕ. The recent results of Fredholm, Hilbert and Schmidt in this direction are fundamental for the standpoint and theorems of the paper.

11. Beginning with the idea of $n + 2$ points in S_n associated with $n + 2$ S_{n-1}'s, one may define a $C^{n+1}_{n+2, r}$ associated with a

given $C^n_{n\cdot 2, r}$ (the notation being that used by Dr. Carver in an earlier paper, "On the Cayley-Veronese class of configurations," *Transactions*, October, 1905). A method is given for constructing the polar point of a $C^{n-1}_{n\cdot 1,}$ with respect to a $C^n_{n+1, r}$; and also for constructing a polar S_{k-1} of a $C^{n-k}_{n+1, r}$ with respect to a $C^n_{n\cdot 1, r}$.

12. In his first paper, Professor Dickson considers commutative linear algebras in $2n$ units, with coördinates in a general field F, such that n of the units define a subalgebra forming a field $F(J)$. The elements of the algebra may be exhibited in the form $A + BI$, although the algebra is not binary. Algebras VIII and X, on four and six units, respectively, given in the July number of the *Transactions*, may now be given a more luminous form (valid for n arbitrary), in which the product of any two elements is

$$(A + BI)(X + YI) = AX + B'Y'J + (AY + BX)I,$$

where $B' = B(J')$, J' being a second root of the normal equation satisfied by J. If the constant term of the latter equation is a not-square in F, division is uniquely possible in the algebra.

By an investigation of the general commutative linear algebra in $2n$ units in which division is unique, we are led to algebras with the law of multiplication $(A + BI)(X + YI) = R + SI$, where

$$R = AX + L(BY - B'Y') + JB'Y',$$

$$S = AY + BX + U(BY - B'Y').$$

For $n = 2$ and $J^2 - C_1J + C_2 = 0$ the conditions for unique division are

$$L = J^2U'^2/4C_2 - \tfrac{1}{4}U^2, \quad C_2 = \text{not-square in } F,$$

$$(J'U^2 + JU'^2 - 4C_2)^2 - 4C_2U^2U'^2 = \text{square in } F.$$

Aside from the case $U = 0$ (giving the first algebra above), these algebras are equivalent, under linear transformation, in sets of four, while the identity is the only transformation into itself of such an algebra. The article will appear in the *Transactions*.

13. In his second paper, Professor Dickson defines abstractly various systems of linear groups with coëfficients in an arbitrary field F, viz., the general linear group, the abelian and hyperabelian groups, and the systems of linear groups defined by a quadratic or a hermitian form. The simplicity of the sets of generational relations is due to the choice of the generators (all of period p when F has modulus p; all non-periodic when F is non-modular). The great majority of the relations express the commutators of each pair of generators. The defining relations have intimate, but not entirely obvious, connections with Lie's commutator relations for the corresponding continuous groups. Two papers incorporating the results for the general linear and the abelian groups have been offered for publication in the *Quarterly Journal of Mathematics.*

16. A systematic study of the geometry of the general vector field without immediate reference to physical applications would be of considerable mathematical interest, and, if carried far enough, must ultimately throw new light on various physical problems. In Professor McMahon's preliminary paper, the vector lines of the field are viewed as a family of twisted curves, which may or may not have a family of orthogonal surfaces. The jacobian determinant of u, v, w, the three rectangular components of the field at (x, y, z), is studied as a fundamental invariant. The linear transformation of any vector (l, m, n), by using a matrix composed of the elements of the jacobian of (u, v, w), produces another vector that bears important relations to the field. The conjugate and reciprocal matrices produce other transformed vectors whose invariant relations are studied, and interpreted geometrically. The differential equations of various other invariant lines are derived, such as the locus of inflexions, lines along which the field vector preserves its direction or magnitude, points of maximum curvature, direction of greatest and least derivative of vector or of tensor. The properties of the auxiliary quadrics and of the general vector tube are considered.

18. Dr. Sisam discusses in this paper the properties of algebraic surfaces generated by systems of conics. A number of general properties of such surfaces are determined and a classification is made of the surfaces so generated which are of order not higher than five.

19. The purpose of Professor Snyder's paper was to find those quintic curves which are left invariant by linear transformations. The only collineations that can transform a plane quintic into itself must all have the same invariant triangle. Apart from a few obvious types and admitting one cyclic collineation, thirteen forms were found. In particular the triangular-symmetric curve has a group of order 150. The largest number of harmonic homologies in any group is 15; in two different types 5 can exist, and one has 3. The remaining types have not more than one.

20. Professor Mason gave a proof for the expansion of an arbitrary function which has a sectionally continuous first derivative in terms of normal functions of the differential equation

$$\frac{d^2y}{dx^2} + \lambda A(x)y = 0.$$

Application was made of the fact that the normal functions are solutions of certain minimum problems.

21. In Professor Mason's second paper a boundary value problem for the differential equation

$$\frac{\partial^2 u}{\partial x^2} = a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu + f$$

was treated which includes several of the boundary problems previously considered. The resolution of the problem was made to depend on an indefinite integral equation, the solution being obtained by successive substitutions.

22. The determination of the trajectories of a given positional force requires the integration of the equations of motion. The inverse problem which forms the subject of Professor Kasner's paper consists in determining the field of force when the trajectories are given. Analytically the problem is solved by a quadrature. The object of the paper is to obtain a direct geometric construction of the field. This is attained by employing the auxiliary system of curves, termed velocity curves, which the author has studied in a previous paper. This related system is decomposable in a simple way into ∞^1 subsystems, from any

one of which the field can be constructed. The problem is solved for both plane and space fields.

23. In previous papers Professor Kasner has studied the motion of a particle acted upon by an arbitrary (positional) field of force in the plane. A complete set of geometric properties of the trajectories is presented in the *Transactions* for July, 1906. The present paper discusses the general three-dimensional problem. The results are not only more complicated, as is to be expected, but also quite different in character. The osculating spheres of the twisted curve take the place of the osculating parabolas employed in the plane theory to interpret differential elements of the third order. It is shown that the osculating spheres of the ∞^1 trajectories passing through a given point in a given direction have their centers on a straight line; and that, as the initial direction varies, the straight line generates a congruence consisting of the secants of a twisted cubic. Converse questions are considered and ultimately a characteristic set of properties is obtained.

24. Professor Young's paper considers a general problem of approximation, which he states as follows: Given a class $\mathfrak{C}_n$ of functions $f=f(x)=f(x;\ a_0, a_1, \cdots, a_n)$ of the real variable x and a functional operation U, such that every $U(f)$ is a fully defined function of x at every point of a finite interval (a, b) as soon as the parameters a_i are given; to determine these a_i so that the maximum of $|U(f)|$, as x varies over (a, b), shall be as small as possible. The existence of a solution, which is called a "function of approximation in $\mathfrak{C}_n$ with reference to U," is proved under very general conditions on $U(f)$. The author then assigns to the functions f a more definite form; *i. e.*, he considers a class $\mathfrak{S}_n$ of functions

$$S_n = a_0 s_0 + a_1 s_1 + \cdots + a_n s_n,$$

where the s_i are given functions of x, and seeks to determine the a_i so that the minimum of $|V(S_n) - \phi|$, as x varies over (a, b), shall be as small as possible. Here V is a distributive functional operation and ϕ a given function of x. Under certain general conditions on V, s_i, and ϕ the problem is shown to have a unique solution, and certain necessary and sufficient conditions are derived that a function in $\mathfrak{S}_n$ shall be a "function of approximation in $\mathfrak{S}_n$ with reference to ϕ and V." The theorems thus proved form a two-fold generalization of the theorems of Tchebychev considered by the author in a paper presented to the Society at its last April meeting, and show that the properties expressed in the older theorems are characteristic of a much larger class of problems than those to which they originally applied.

25. The paper by Professor Hutchinson considers the loci in space of four dimensions determined by taking the five homogeneous coördinates of a point proportional to linearly independent theta functions of three variables of the same order and characteristic, and all odd, or all even. It is found that only a finite number of such loci exist having a given degree. By equating to zero a linear combination of the given theta functions, algebraic surfaces in space of three dimensions are determined. Various results are deduced by means of the properties of the theta functions.

26. Professor Eisenhart considers pairs of surfaces applicable to one another with the asymptotic lines of one surface corresponding to a conjugate system on the other. He finds that each of the two surfaces is an associate of a spherical surface— a surface whose gaussian curvature is positive and constant— and the two spherical surfaces are the Hazzidakis transforms of one another. Moreover, every surface associate to a spherical surface is applicable to a surface associate to the Hazzidakis transform of the spherical surface, in such a way that the asymptotic lines of the one correspond to a conjugate system of the other. To asymptotic lines on the associate surface corresponds a conjugate system with equal tangential invariants on the given spherical surface, and on its Hazzidakis transform a conjugate system with equal point invariants. When a conjugate system upon a spherical surface has equal point and tangential invariants, two pairs of applicable surfaces of the kind sought can be found by quadratures; a particular study is made of this case. When a surface is referred to its asymptotic lines, the equations of condition to be satisfied in order that the surface admit an applicable surface upon which the parametric lines form a conjugate system are of such a form that it is seen that if there are more than two such applicable surfaces there are an infinity. In this case the fundamental coefficients of the applicable surfaces can be found directly and the infinity of associate spherical surfaces are given by quadratures.

27. In the *Mathematische Annalen* (volume 39 (1891), pages 324–326) Scheffers calls the algebra $EF \equiv \epsilon_{ij} = e_i f_j = f_j e_i$ $(i = 1, \cdots, n; j = 1, \cdots, r)$ the product (compound) of the algebras $E \equiv e_1 \cdots e_n$, $F \equiv f_1 \cdots f_r$. He suggests inversely the desirability of theorems concerning the roots of the characteristic equation of the composite algebra, of a criterion for determining whether a given algebra is composite, and also of theorems concerning the division of zero in the composite algebra.

These problems are studied in a thesis prepared by Mr. Leonard in candidacy for the degree of doctor of philosophy from the University of Colorado. In § 1 it is shown that if G_E and G_F are the groups of the algebras E and F, then the transverse of the compound of these groups is a subgroup of the group G_{EF} of the compound algebra EF. In § 2 the first of the above problems is treated and the following theorem proved: If $\mu_1, \cdots, \mu_n$ are the roots of the characteristic equation of the number $X = \sum_i x_i e_i$ of the algebra E and if $\nu_1, \cdots, \nu_r$ are the roots of the characteristic equation of the number $\bar{X} = \sum_j \bar{x}_j f_j$ of the algebra F, then the roots of the corresponding number of the compound algebra are $\mu_i \nu_j (i = 1, \cdots, n; j = 1, \cdots, r)$.

The second of the above problems is treated from two distinct points of view. In § 3 the characteristic equation of the composite algebra EF being given, the characteristic equations of the factor algebras are sought. In § 4 the multiplication table of the composite algebra being given, the multiplication tables of the factor algebras are determined by means of Peirce's matrix representation of linear associative algebras. For example, by both methods the algebra

$$\begin{array}{c|cccc}
 & \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 \\
\hline
\epsilon_1 & \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 \\
\epsilon_2 & \epsilon_2 & 0 & \epsilon_4 & 0 \\
\epsilon_3 & \epsilon_3 & \epsilon_4 & -\epsilon_1 & -\epsilon_2 \\
\epsilon_4 & \epsilon_4 & 0 & -\epsilon_2 & 0
\end{array}$$

is factored into the algebras

$$\begin{array}{c|cc}
 & e_1 & e_2 \\
\hline
e_1 & e_1 & e_2 \\
e_2 & e_2 & 0
\end{array}
\qquad
\begin{array}{c|cc}
 & f_1 & f_2 \\
\hline
f_1 & f_1 & f_2 \\
f_2 & f_2 & -f_1
\end{array}$$

Thus one comes to the interesting classification of algebras into prime and composite, exactly like ordinary integers, and this leads to a theory of linear associative algebras analogous to the theory of integers.

If one of the factors is a divisor of zero, it is shown in § 5 that the composite number is a divisor of zero, and conversely.

28. Professor Morley's paper on reflexive geometry is in continuation of two which appeared in the *Transactions*, volumes 1 and 4. Given n lines of a plane, a certain curve C^{n-1} is thereby given, and the discussion of this curve in relation to the lines is the main theme. The curve C^2 is a circle, the curve C^3 a cardioid. The C^{m-1} of any m of the n lines is an osculant of the C^{n-1} of all the lines. If we build up all Jonquières curves of order $n - 2$, with multiple point of order $n - 3$ at a point x, and passing through the reflexions of x in the n lines, then such a curve has (projectively speaking) coincident points at infinity when x is on C^{n-1}, and it has equiangular asymptotes when x is on the centric circle. Incidentally it appears that n lines belong to and define a set of $2n - 3$ lines. The case when for n odd the n lines form such a set is exceptional.

29. The object of Professor Miller's paper is to study all the groups which result if the defining equations of the groups of genus zero are generalized along certain lines. It is a continuation of the paper in volume 9 of the *Archiv der Mathematik und Physik* under the heading "The groups generated by two operators which have a common square." Some of the results are as follows: There are exactly four groups whose two generators s_1, s_2 satisfy the relations $s_1^3 = s_2^3$, $(s_1s_2)^2 = 1$. They are the tetrahedral group, the direct product of this group and the cyclic group of orders 4 and 2, and a certain group of order 96. There are also just four groups whose two generators satisfy the conditions $s_1^2 = s_2^3$, $(s_1s_2)^3 = 1$, while there is an infinite system of groups which satisfy the conditions $s_1^3 = s_2^3$, $(s_1s_2^{-1})^2 = 1$. The similar generalizations of the defining equations of the octahedral and the icosahedral groups are considered and all the possible groups are completely determined. The paper has been offered to the *Transactions* for publication.

30. In the paper on divergence and curl Professor Wilson starts with the integral definitions of these differentiators. This

leads to a closer contact with physics and hydrodynamics, and brings out almost intuitively the theorems of Gauss and Stokes. The introduction of the linear vector function $\nabla \mathbf{V}$ (where $\mathbf{V}$ is a vector function of position in space) makes it possible to obtain div $\mathbf{V}$ and curl $\mathbf{V}$ as $\nabla \cdot V$ and $\nabla \times V$ respectively without the usual recourse to integral calculations relative to an infinitesimal cube or rectangle. The paper appears in the *American Journal of Science*, 1906.

31. After a short introduction giving the history of the efforts to study various subgroups of the projective group by means of involutory transformations, Professor Wilson proceeds to a systematic study of involutory projective transformations and to their special cases in various subgroups. An algebraic interest is added to the discussion by pointing out the fact that these transformations may be regarded as square roots of the idemfactor. In the main body of the paper the discussion of the composition of oblique reflection into strains and the resolution of unimodular strains into reflections is treated in detail. The analytic method employed is Gibbs's vector analysis—in particular his theory of dyadics. Among other things it is proved that: Any root of the minor idemfactor may be resolved into the product of two square roots in ∞^1 ways. The necessary and sufficient condition that a strain be resolved into two reflections is that the scalar cubic corresponding to the identical equation in the matrix representing the strain shall be a reciprocal equation. Any unimodular strain may be resolved into three reflections and the line of the first reflection is arbitrary except possibly for at most three directions.

32. In the paper on double products Professor Wilson takes up somewhat at length the generalizations of some of the questions treated in chapters V and VI of his edition of Gibbs's Vector Analysis. The present paper is perhaps the first published account of the late Professor Gibbs's ideas on multiple algebra, except for the general papers "On multiple algebra" by J. Willard Gibbs, *Proceedings of the American Association for the Advancement of Science*, volume 35 (1886), and "On products in additive fields" by the present author, Verhandlungen des III internationalen Mathematiker-Kongresses in Heidelberg, 1904. The paper will be offered for publication in the *Transactions*.

33. The paper of Professor Moulton is devoted to the consideration of a class of periodic orbits in which the mean motion of the line of nodes of the orbit of one the bodies, referred to the mean plane of the motion of the other two, is arbitrary except for the condition that it shall be commensurable with the synodic mean motion of the three bodies. The discussion involves a new treatment of Hill's linear differential equation with periodic coefficients.

As applied to the lunar theory, the paper shows how to construct expressions for the coordinates of a body having the same synodical period and mean rate of revolution of the line of nodes as observations show the moon has. These are the terms which are said to depend upon the mean motions of the sun and moon, the parallax of the moon, and the latitude of the moon.

34. Professor Bolza's paper gives an extension of Weierstrass's theorem on the expression of the total variation by means of the E-function and of Kneser's theorem on transversals to the so-called most general case of an extremum of a simple definite integral, in which it is required to minimize an integral of the form

$$I = \int_{x_0}^{x_1} f(x, y_1, \cdots, y_n; y_1', \cdots, y_n')dx$$

involving n unknown functions $y_1, \cdots, y_n$ of x and their first partial derivatives $y_1', \cdots, y_n'$, connected by $r < n$ differential equations $f_\rho(x, y_1, \cdots, y_n; y_1', \cdots, y_n') = 0$ $(\rho = 1, 2, \cdots, r)$.

F. N. Cole,
Secretary.

THE NEW HAVEN COLLOQUIUM.

The Fifth Colloquium of the American Mathematical Society was held, at the close of the thirteenth summer meeting, at Yale University, New Haven, Conn., opening on Wednesday morning, September 5, 1906 and extending until noon of the following Saturday. Since the colloquium has become a highly important element in the Society's activities, an outline of its historic development may here be of interest.

The first colloquium organized by the Society was held in connection with the third summer meeting at Buffalo, N. Y., September 2–5, 1896. Two courses of six lectures each were delivered before an audience of thirteen members. Professor Maxime Bôcher considered "Linear differential equations and their applications," and Professor James Pierpont discussed "The Galois theory of equations."*

The second colloquium was held at the close of the fifth summer meeting, at Cambridge, Mass., August 22–27, 1898, before twenty-six persons. Professor W. F. Osgood delivered six lectures on "Selected topics in the general theory of functions," and Professor A. G. Webster six lectures on "The partial differential equations connected with wave propagation."†

The third colloquium, held in connection with the eighth summer meeting, at Ithaca, N. Y., August 21–24, 1901, was attended by twenty-five persons. Two courses of four lectures each were delivered: by Professor Oskar Bolza on "The simplest type of problems in the calculus of variations," and by Professor E. W. Brown on "Modern methods of treating dynamical problems and in particular the problem of three bodies."‡

The fourth colloquium was held immediately after the tenth summer meeting at Boston, Mass., September 2–5, 1903, before an audience of thirty-one members. Three courses of lectures were given: by Professor E. B. Van Vleck, "Selected topics in the theory of divergent series and continued fractions," six lectures; Professor H. S. White, "Linear systems of curves on algebraic surfaces," three lectures; Professor F. S. Woods, "The connectivity of non-euclidean space," three lectures.§

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*See the report by Professor T. S. Fiske, BULLETIN, series 2, volume 3, pages 49–59. Professor Bôcher's lectures were published in part in the *Annals of Mathematics*, series 1, volume 12 (1898), pages 45–53. Professor Pierpont's lectures were published in the same journal, series 2, volume 1 (1899), pages 113–143 and volume 2 (1900), pages 22–56.

†See the report by Professor H. S. White, BULLETIN, volume 5, pages 57–58. Professor Osgood's lectures were published in the same volume, pages 59–87. The lectures of Professor Webster are to be published soon.

‡See the report by Professor Edward Kasner, BULLETIN, volume 8, pages 22–25. Professor Bolza's lectures are embodied in his Lectures on the Calculus of Variations, University of Chicago Decennial Publications, No. 14. An abstract of Professor Brown's lectures was published in the BULLETIN, volume 8, pages 103–113.

§See the report by Professor F. N. Cole, BULLETIN, volume 10, pages 119–120, and an abstract of Professor White's lectures immediately following. All the courses have since been published for the Society by the Macmillan Company under the title The Boston Colloquium. This book is reviewed by Professor J. I. Hutchinson in the present number of the BULLETIN.

At the December meeting 1905, the Council appointed a committee consisting of Professors J. Pierpont, P. F. Smith, H. Maschke, H. S. White and F. N. Cole to arrange for a fifth colloquium. A preliminary circular announcing the general features was issued in June. The colloquium opened on Wednesday morning, September 5, 1906, in the lecture rooms of Lampson Hall, Yale University, the following 43 persons being in attendance:

Mr. G. D. Birkhoff, Professor G. A. Bliss, Professor E. W. Brown, Dr. A. B. Chace, Dr. J. E. Clarke, Dr. A. Cohen, Mr. H. H. Conover, Professor D. R. Curtiss, Mr. H. N. Davis, Professor L. P. Eisenhart, Professor W. B. Fite, Professor A. S. Gale, Professor C. N. Haskins, Dr. L. I. Hewes, Professor Edward Kasner, Professor O. D. Kellogg, Dr. W. R. Longley, Mr. E. B. Lytle, Professor James McMahon, Professor T. E. McKinney, Professor H. P. Manning, Professor Max Mason, Mr. E. A. Miller, Professor E. H. Moore, Professor James Pierpont, Professor H. L. Rietz, Dr. R. G. D. Richardson, Miss S. F. Richardson, Mr. A. R. Schweitzer, Miss I. M. Schottenfels, Professor C. S. Slichter, Dr. Clara E. Smith, Professor P. F. Smith, Professor Virgil Snyder, Professor H. F. Stecker, Dr. R. P. Stephens, Professor W. E. Story, Professor E. B. Van Vleck, Professor A. G. Webster, Professor H. S. White, Professor E. J. Wilczynski, Professor T. W. D. Worthen, Professor J. W. Young.

Three courses of lectures were given, as follows:

Professor E. H. MOORE: "On the theory of bilinear functional operations." Five lectures.

Professor E. J. WILCZYNSKI: "Projective differential geometry." Four lectures.

Professor MAX MASON: "Selected topics in the theory of boundary value problems of differential equations." Four lectures.

Two lectures were given each morning, two on each of the afternoons of Wednesday and Friday, and one on Wednesday evening, the lecturers alternating. Printed syllabi of all the courses had been issued in advance, which greatly aided intelligent participation. Thursday afternoon was devoted to a trip on Long Island Sound; in the evening the members dined together at Momauguin. The remaining evenings were devoted to social conference at the Graduate Club, whose privileges were extended to members of the Society. The hospitality of the Club, of Yale University, and particularly of the members of the mathematical department were gratefully acknowledged by a unanimous vote of thanks and appreciation at the closing meeting.

Detailed reports of the courses, prepared by the lecturers, will appear in later numbers of the BULLETIN.

VIRGIL SNYDER.

THEORY AND CONSTRUCTION OF TABLES FOR THE RAPID DETERMINATION OF THE PRIME FACTORS OF A NUMBER.*

BY PROFESSOR ERNEST LEBON.

BY making use of some hitherto unnoticed properties of certain arithmetic progressions, I have succeeded in constructing a table giving very rapidly the solution of the following double problem: *To determine whether a given number is prime or composite, and in the latter case to find its prime factors.* The process which I employ is applicable to large numbers.†

1. Let B be the product $\alpha\beta\cdots\lambda$ of the consecutive prime numbers $\alpha, \beta, \cdots, \lambda$, beginning with 2; P the product $(\alpha-1)(\beta-1)\cdots(\lambda-1)$; I any of the P numbers that are relatively prime to B and less than B; K a number successively equal to the positive integers, starting from zero.

We easily see that *the system of P arithmetic progressions whose general term is $BK+I$ contains all the prime numbers except those that occur in B.*

We shall say that B is the base of the system and that I is the index of a term of this system.

Two indices will be said to be complementary when their sum is equal to the base.

2. Let N, D and M be any numbers relatively prime to B. In order to avoid ambiguity, I will write D in the form $BK'+I'$.

It is evident that $N(=BK+I)$ *is or is not divisible by D according as K and M do or do not satisfy the equation*

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* Translated by Professor W. B. FITE.

† Cf. *Comptes rendus*, vol. 151 (1905), p. 78. See also § 10, p. 77.

$$BK + I = MD, \tag{a}$$

B, I and D being known.

3. Let k and m be the minimum values of K and M satisfying equation (a), and n a number successively equal to the positive integers starting from zero. If necessary for clearness, I use k_I for the numbers K relative to a divisor D.

The equality

$$K = k + nD$$

gives the values of K to which correspond all the numbers N that are divisible by D.

From this equality we get the formula

$$n = \frac{K - k}{D}, \tag{1}$$

where K is the integral quotient obtained by dividing N by B; the remainder in this division is the value of I.

We see that *according as the value of n obtained by applying formula* (1) *is integral or fractional, the number N is, or is not, a multiple of the divisor D.*

Then *the table of numbers k set up for a system of base B enables one to recognize whether N is prime or not by dividing the difference K — k by the prime numbers less than $\sqrt{N}$ and greater than λ; if N is not prime, this procedure gives its prime factors.*

We see that the larger the base B the more rapidly this method gives the result.

Before applying formula (1), it should not be forgotten that if we are considering a number N' we must in order to get N remove from it the factors that are common to it and B.

4. The numbers k I shall call *characteristics.*

5. In order to find methodically and quickly the characteristics k which correspond to the P arithmetic progressions of a system with the base B, we can use the following formula, which is obtained by replacing in equation (a) K and M by k and m, and D by $BK' + I'$:

$$k = \frac{I'm - I}{B} + K'm. \tag{2}$$

Formula (2) gives the characteristic k when the value of m is such that the binomial $I'm - I$ is divisible by B.

6. The three following theorems, which are easily demonstrated, enable one to make a considerable reduction in the number of operations required for the calculation of the characteristics k:

I. *To the product* $I'm$ *of the two indices* I' *and* m *correspond an index* I *and a characteristic* k; *this characteristic is associated with the number* $I'm$ *by the arithmetic progression of base* B *and index* I *given by this product.*

II. *The* P *arithmetic progressions of a system of base* B *being arranged in the order of the increasing values of the indices* I *of their terms, the sum of the two characteristics* k *and that of the two values of* m *relative to the same divisor* D *and to two progressions equidistant from the extremes are equal to* $D - 1$ *and* B *respectively.*

III. *If the values of* I, k_I, I', *and* m *satisfy the equation*

$$Bk_I + I = I'm,$$

and if we consider the equation

$$Bk_{B-I} + (B - I) = (B - I')m,$$

where the two indices $B - I$ *and* $B - I'$ *are complementary to the* I *and* I' *respectively of the preceding equality, the unknown characteristic* k_{B-I} *is given by the formula*

$$k_{B-I} = m - 1 - k_I.$$

7. It follows from Theorems II and III that in order to calculate the binomial $I'm - I$, it is sufficient to associate with the first half of the P values of I' the first half of the P values of m, arranged in the order of magnitude.

The remainder obtained by dividing $I'm$ by B is the index I relative to a progression of the system of base B.

When K' is zero, the first term of formula (2) gives, in each of the P progressions of base B, the P characteristics k corresponding to the P values of I.

Inasmuch as the characteristics k corresponding to the index I are the same when D is equal to either I' or m, it follows from Theorem I that it is sufficient to take the products $I'm$

starting from the value of m equal to the value of I'; that is to say, it is sufficient to take the values of $I'm$ starting from I'^2. We know that we apply the first term of formula (2) only to the values of m which are equal to the first $P/2$ indices. Moreover to the products of 1 by the indices correspond characteristics k which are evidently zero.

Consequently, among the P^2 characteristics k relative to the P divisors which equal the indices there are at most $P(P-2)/8$ characteristics whose determination requires a multiplication and a division.

8. As to the P characteristics k relative to a divisor D superior to $B-1$ and with index I', we can deduce them immediately from the P characteristics found for $D=I'$ by making use of the last term of formula (2).

9. In order to apply formula (1), we can make use of a table of characteristics relative to the base B containing at the top of the columns only the first half of the P indices I arranged in order of magnitude, and below each index I_n the complementary index $B-I_n$; then in these columns, in regard to the prime divisors D, the values of k relative to the first half of the P indices I.

Then, having a number N which does not contain any of the prime factors of B, we divide N by B. This gives the quotient K and the remainder I, which I shall call I_n if it belongs to the first half of the P indices I arranged in order of magnitude, and $I_{n'}$ if it belongs to the second half of these indices.

When the remainder is I_n, the index is also I_n and the characteristic k is equal to the value k_n given in the table.

According as D is, or is not, a multiple of the difference $K-k_n$, D is, or is not, a prime divisor of the number whose index is I_n or of the number whose index is $I_{n'}$.

10. The table of characteristics relative to the base 30030, with the prime divisors from 17 to 30029 enables one to solve the problem in question between 1 and 30030^2 or 901800900.

Suppose that the table of characteristics k relative to the base B is formed of columns headed by all the indices I in order of magnitude and of rows headed by the prime divisors D arranged in the order of magnitude. The characteristic k corresponding to a number N of index I_n and to a prime divisor D is found at the intersection of the column I_n and the row D.

Let N be a number of the form $30030K+I$. In making the trial we will stop at the prime divisor D_n immediately inferior to $\sqrt{N}$.

We consider whether K is equal to one of the characteristics which correspond to the index I; for this it is sufficient to start from the prime divisor immediately superior to K.

When K is equal to one or several of these characteristics, N admits prime divisors which correspond to these characteristics. Then we have immediately the composition of N.

When K is not equal to any of these characteristics, we form the differences $K - k$ for the prime divisors 17, 19, 23, $\cdots$. These differences are always less than 30029, because K is here less than B and k is less than D and hence less than B. A difference $K - k$ is, or is not, equal to an index. In the former case, we recognize without calculation whether the difference $K - k$ is divisible by the divisor that corresponds to it. In the latter case we recognize nearly always whether a difference $K - k$ is divisible by the corresponding divisor D without performing the division; then we decompose $K - k$ into factors, one of which is either one of the prime numbers 2, 3, 5, 7, 11 and 13, or a product of some of these, and the other an index. In most cases it is not necessary to perform this decomposition in order to see if a difference is divisible by the prime divisor which corresponds to it.

If there is no difference $K - k$ that is divisible by any of the prime factors from 17 to D_n, N is prime. If we find a difference $K - k$ that is divisible by the prime divisor D less than D_n, N is divisible by D. We divide N by D, the resulting quotient also by D, and so on. Let N_1 be the last quotient thus obtained. We treat N_1 as we have just treated N, beginning with the prime divisor immediately following D, and we find that N_1 is the product of characteristics or is prime.

11. In order to recognize instantly whether a difference $K - k$ is divisible by the corresponding divisor D, it is sufficient to have, in addition to the table of characteristics relative to the base 30030 up to the divisor 30029, a table of remainders obtained by dividing the consecutive integers from 17 to 30029 by the divisors D; in fact, a difference $K - k$ is divisible by the corresponding divisor D, when the values of R and of k which correspond to this division are equal.

PARIS,
May, 1906.

ON A FUNDAMENTAL RELATION IN ABSTRACT GEOMETRY.

BY MR. A. R. SCHWEITZER.

(Read before the American Mathematical Society, September 3, 1906.)

1. In his Abstrakte Geometrie, page 10, Professor Vahlen seeks to define planar order with the aid of two relations, each of which is between a point and a set of linearly ordered points, such that if bc and b_1c_1 are any two pairs of distinct points of the set, b precedes (follows) c and b_1 precedes (follows) c_1 simultaneously. The linear order referred to is established by means of the undefined relation due to Vailati which may be expressed concretely by "precede." Aside from the fact that Vahlen employs a superfluity of undefined symbols, there is a noticeable lack of precision in his presentation.* Recent investigations of the writer have emphasized the importance, in abstract geometry, of a relation used implicitly by Vahlen, which, however, seems to have escaped him entirely. The object of the present note is briefly to point this out.

2. The relation referred to above is $aR(bc)$, which may be considered an abstract statement for the following: "If a person swims from b to c the point a is at his right." In the relation $aR(bc)$, (bc) is an ordered dyad. That is, if $b \neq c$, then the dyads (bc) and (cb) are distinct, and if the points bc are not the points b_1c_1, then the dyads (bc) and (b_1c_1) are also distinct. Thus if (bc) and (b_1c_1) are identical, ordered dyads, then $b = b_1$ and $c = c_1$. The existence of an ordered dyad, it will be observed, does not properly involve any geometric assumptions, except the existence of points.

3. We can now construct planar axioms and definitions which will clarify the Vahlen text.

1. AXIOMS.

(1) There is at least one point.

(2) If there exists a point, there exist the points a_0, b_0, c_0 such that $a_0R(b_0c_0)$ or $a_0R(c_0b_0)$.

(3) If $aR(bc)$, then $a \neq b$, c.

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* Cf. the review by Professor Veblen, BULLETIN, July, 1906, p. 505.

(4) If $aR(bc)$ and $b \neq c$, then $a\bar{R}(cb)$.*

(5) If $aR(bc)$ and $x \neq a$, then $xR(bc)$ or $xR(ca)$ or $xR(ab)$.

2. DEFINITIONS.

(1) The point x is *collinear* with (ab) if $a \neq b$, $x\bar{R}(ab)$, and $x\bar{R}(ba)$.

(2) The point x is in the *interior* of (ab) if
there is a point c' such that $c'R(ab)$,
$x\bar{R}(ab)$, $x\bar{R}(ba)$,
for *any* point c such that $cR(ab)$ and $xR(bc)$ or $xR(ca)$ follows $xR(bc)$, $xR(ca)$.

(3) The point x is in the *plane* (abc) if $aR(bc)$ and either $xR(bc)$ or $xR(ca)$ or $xR(ab)$.

(4) The point x is in the *interior* of (abc) if $xR(bc)$, $xR(ca)$, and $xR(ab)$.

It will be noticed that it is not necessary to introduce an additional relation $aL(bc)$ as Vahlen does. This relation is expressed by $aR(cb)$. The definitions (1)–(3) are not given by Vahlen.

4. As a consequence of the preceding axioms we prove the following (Cf. Vahlen, Axiom, § 21, page 10).

Theorem 1. If $cR(ab)$, then $aR(bc)$ and $bR(ca)$.

Since $cR(ab)$, by axiom (3), $c \neq a, b$. Hence by axiom (5), $aR(ab)$ or $aR(bc)$ or $aR(ca)$. Therefore by (3), $aR(bc)$. Similarly, $bR(ca)$.

Theorem 2. If $xR(ab)$ and $xR(bc)$ and $xR(ca)$, then $cR(ab)$.

We may assume $c \neq x$. Then by axiom (5), $cR(ab)$ or $cR(xa)$ or $cR(bx)$. By theorem 1, if $cR(xa)$ or $cR(bx)$, then $xR(ac)$ or $xR(cb)$ respectively. Hence by axioms (3), (4), and the hypothesis, $cR(ab)$ is the only possibility. The theorem therefore follows.

5. It would be easy to construct further axioms so that the preceding definitions are effective. For this purpose it will be most convenient to refer to the writer's planar descriptive system.† There results, then, that by using two undefined symbols, point and the (planar) relation $aR(bc)$, it is possible to construct a set of axioms for the plane without introducing an undefined linear relation as Vahlen has done. However, the relation $aR(bc)$ or, as we will now write it, $aR_2(bc)$ may be considered an extension, dimensionally, of the undefined relation

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* The rule over the R is a symbol of negation.

† Cf. BULLETIN, June, 1906, p. 438. For n-ad read $(n+1)$-ad.

of Vailati, which we will write aR_1b. More generally, it is possible to construct a set of n-dimensional axioms ($n = 1, 2, 3, \cdots$) by means of the two undefined symbols point and the n-dimensional relation $aR_n(b_1b_2 \cdots b_n)$, where the n-ad is ordered.

CHICAGO,
July, 1906.

ON THE ORDERLY LISTING OF SUBSTITUTIONS.

BY PROFESSOR D. N. LEHMER.

(Read before the San Francisco Section of the American Mathematical Society, February 24, 1906.)

1. In a substitution such as $\begin{pmatrix}12345\\34251\end{pmatrix}$ we shall speak of 34251 as the *permutation* belonging to the substitution. The following list of the permutations belonging to the substitutions on four letters exhibits what seems to be the most orderly arrangement:

$1 = 1234 = I$
$2 = 1243 = (34)$
$3 = 1324 = (23)$
$4 = 1342 = (234)$
$5 = 1423 = (243)$
$6 = 1432 = (24)$
$7 = 2134 = (12)$
$8 = 2143 = (12)(34)$
$9 = 2314 = (123)$
$10 = 2341 = (1234)$
$11 = 2413 = (1243)$
$12 = 2431 = (124)$
$13 = 3124 = (132)$
$14 = 3142 = (1342)$
$15 = 3214 = (13)$
$16 = 3241 = (134)$
$17 = 3412 = (13)(24)$
$18 = 3421 = (1324)$
$19 = 4123 = (1432)$
$20 = 4132 = (142)$
$21 = 4213 = (143)$
$22 = 4231 = (14)$
$23 = 4312 = (1423)$
$24 = 4321 = (14)(23)$

2. It is seen that the numbers of four digits expressed by the different permutations are in ascending order of magnitude. This order would be reversed by the interchange of 1 with 4 and 2 with 3.

3. It is easily possible to compute the rank of a given permutation or to write down a permutation of given rank. Thus the rank of the permutation 341562 is 240 plus the rank

of 41562, there being 120 beginning with 1 and 120 beginning with 2. The rank of 41562 is 48 plus the rank of 1562, since there are 24 beginning with 1 and 24 beginning with 2. The rank of 1562 is the same as the rank of 562 or, since our list gives the rank of any permutation on four elements, we can obtain the rank of 1562 from it as 4. The rank of 341562 is therefore 292. The finding of a permutation of given rank is not more difficult.

4. The permutation of rank 292 in seven letters is 1452673 obtained by prefixing unity and increasing each digit by unity. Similarly for eight letters the permutation of rank 292 is 12563784. The cyclic notations for the corresponding substitutions are

For six letters (13)(2456),
For seven letters (24)(3567),
For eight letters (35)(4678).

In accordance with this rule the multiplication table for the symmetric group on n letters will contain in the upper left hand corner the multiplication table for the symmetric group on $n-1$ letters.

5. The product of two substitutions S and S' in the order SS' is obtainable by operating on the permutation of S with the substitution S', as is readily seen by changing the order of the elements in the upper line and accordingly in the lower line of S'. This law enters in a remarkable way in the multiplication table of the symmetric group. One obvious result is seen in the last column under the substitution (14) (23). This, as we noted in § 2, reverses the order of the permutations so that we have in that column the substitutions in descending rank reading downward. In symbols $S_iS_{24} = S_{25-i}$, or since the permutation of rank $n!$ reverses the order of the arrangement we have generally

$$S_iS_{n!} = S_{n!-i+1}.$$

6. Suppose now that $S_iS_j = S_k$. Then

$$S_iS_jS_{n!} = S_kS_{n!},$$

whence by the above formula

$$S_iS_{n!-j+1} = S_{n!-k+1}.$$

From this it follows that two entries in the table which are on the same horizontal line and are equally distant from the ends add up $n! + 1$. Thus $12 \cdot 17 = 21$ and $12 \cdot 8 = 4$ and $21 + 4 = 25 = 4! + 1$.

7. Suppose again that

$$S_{n!}S_j = S_k.$$

Then

$$S_iS_{n!}S_j = S_iS_k,$$

or

$$S_{n!-i+1}S_i = S_jS_k.$$

From this formula it follows that the column in the table with k for its lowest element reads upward in the same way that the column with k for its topmost element reads downward. These two curious laws hold in their entirety for any subgroup of the symmetric group which contains the substitution of rank $n!$ This in particular will be the case with the alternating group when n is of the form $4m$ or $4m + 1$, since in these two cases the substitution $S_{n!}$ is an even one. The metacyclic group of order $p(p - 1)$ will also exhibit these laws since it contains the substitution $\binom{x_z}{x_{-z+1}}$ which is of rank $n!$

8. The list of the permutations of the substitutions on n letters divides itself naturally into n sets. The first set comprises those beginning with the element 1, the second those beginning with the element 2 and so on. These sets we will call the primary sets. Each of these primary sets is divided into secondary sets $n - 1$ in number, each secondary set beginning with the same two numbers. These secondary sets divide up further into ternary sets and so on. We will examine the arrangement of these sets in the multiplication table of the symmetric group.

9. In the product SS' let S' remain fixed while S runs through the list of $n!$ substitutions in order. This by § 5 amounts to operating on all the permutations in order with the substitution S'. It is clear that none of the sets are separated by this operation, the substitution S' merely interchanging the sets as wholes and permuting the subsets among themselves. Moreover the permutation of the primary sets will be of the same rank as the permutation of S'. Thus in the column headed 10 in the group on four elements it is seen that the primary sets are listed reading down in the order 2341 which is the permutation of rank 10.

10. Consider now any primary set beginning with the element k. The secondary sets in it are clearly permuted among themselves in accordance with the effect of S' on all the elements except k. The permutation of the secondary sets may then be obtained by striking out k from the permutation of S'. Thus in column 10 as before referred to the order of the secondary sets in the first primary set is 341; in the second primary set is 241; in the third primary set is 231, and in the fourth 234. The order for the ternary sets is similarly obtained so that for instance the ternary sets in the first primary set have the orders 41, 31, 34, obtained by striking out 3, 4 and 1 in succession from the order of the secondary set 341.

The multiplication table for the symmetric group on four letters is appended; also the table for the alternating group.

SYMMETRIC GROUP.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15	18	17	20	19	22	21	24	23
3	5	1	6	2	4	9	11	7	12	8	10	15	17	13	18	14	16	21	23	19	24	20	22
4	6	2	5	1	3	10	12	8	11	7	9	16	18	14	17	13	15	22	24	20	23	19	21
5	3	6	1	4	2	11	9	12	7	10	8	17	15	18	13	16	14	23	21	24	19	22	20
6	4	5	2	3	1	12	10	11	8	9	7	18	16	17	14	15	13	24	22	23	20	21	19
7	8	13	14	19	20	1	2	15	16	21	22	3	4	9	10	23	24	5	6	11	12	17	18
8	7	14	13	20	19	2	1	16	15	22	21	4	3	10	9	24	23	6	5	12	11	18	17
9	11	15	17	21	23	3	5	13	18	19	24	1	6	7	12	20	22	2	4	8	10	14	16
10	12	16	18	22	24	4	6	14	17	20	23	2	5	8	11	19	21	1	3	7	9	13	15
11	9	17	15	23	21	5	3	18	13	24	19	6	1	12	7	22	20	4	2	10	8	16	14
12	10	18	16	24	22	6	4	17	14	23	20	5	2	11	8	21	19	3	1	9	7	15	13
13	19	7	20	8	14	15	21	1	22	2	16	9	23	3	24	4	10	11	17	5	18	6	12
14	20	8	19	7	13	16	22	2	21	1	15	10	24	4	23	3	9	12	18	6	17	5	11
15	21	9	23	11	17	13	19	3	24	5	18	7	20	1	22	6	12	8	14	2	16	4	10
16	22	10	24	12	18	14	20	4	23	6	17	8	19	2	21	5	11	7	13	1	15	3	9
17	23	11	21	9	15	18	24	5	19	3	13	12	22	6	20	1	7	10	16	4	14	2	8
18	24	12	22	10	16	17	23	6	20	4	14	11	21	5	19	2	8	9	15	3	13	1	7
19	13	20	7	14	8	21	15	22	1	16	2	23	9	24	3	10	4	17	11	18	5	12	6
20	14	19	8	13	7	22	16	21	2	15	1	24	10	23	4	9	3	18	12	17	6	11	5
21	15	23	9	17	11	19	13	24	3	18	5	20	7	22	1	12	6	14	8	16	2	10	4
22	16	24	10	18	12	20	14	23	4	17	6	19	8	21	2	11	5	13	7	15	1	9	3
23	17	21	11	15	9	24	18	19	5	13	3	22	12	20	6	7	1	16	10	14	4	8	2
24	18	22	12	16	10	23	17	20	6	14	4	21	11	19	5	8	2	15	9	13	3	7	1

ALTERNATING GROUP.

1	4	5	8	9	12	13	16	17	20	21	24
4	5	1	12	8	9	16	17	13	24	20	21
5	1	4	9	12	8	17	13	16	21	24	20
8	13	20	1	16	21	4	9	24	5	12	17
9	17	21	5	13	24	1	12	20	4	8	16
12	16	24	4	17	20	5	8	21	1	9	13
13	20	8	21	1	16	9	24	4	17	5	12
16	24	12	20	4	17	8	21	5	13	1	9
17	21	9	24	5	13	12	20	1	16	4	8
20	8	13	16	21	1	24	4	9	12	17	5
21	9	17	13	24	5	20	1	12	8	16	4
24	12	16	17	20	4	21	5	8	9	13	1

THE BOSTON COLLOQUIUM.

Lectures on Mathematics. By E. B. VAN VLECK, H. S. WHITE, F. S. WOODS. New York, The Macmillan Company, 1905. xii + 187 pp.

THIS volume contains the lectures delivered at the Boston Colloquium of the AMERICAN MATHEMATICAL SOCIETY, September 2–5, 1903; it is not only a convenient form in which to preserve these valuable lectures, but is a most welcome addition to the meagre catalogue of English works on the higher mathematics. In the brief space of 166 pages is given a very clear and readable summary of three important fields of recent development. The topics treated have the following titles:—

Linear systems of curves on algebraic surfaces, by Professor White, pages 1–30;

Forms of non-euclidean space, by Professor Woods; pages 31–74;

Selected topics in the theory of divergent series and of continued fractions, by Professor Van Vleck, pages 76–166, followed by a bibliography of memoirs relating to algebraic continued fractions, pages 167–187.

The authors have displayed admirable judgment in selecting from the abundance of material at their disposal those theorems and results most suited to give a distinct and vivid picture of the fields with which it is intended to make the reader acquainted. A judicious and carefully considered reserve is exercised in the presentation of details, some of the theorems being demonstrated in full, while for others only an outline is given sufficient to present a clear idea of the argument and the essential nature of the method, and still other results are merely stated when their character and bearing can be made evident without the details of proof. The value and suggestiveness of the lectures are augmented by the various critical remarks scattered throughout the work, which serve not only to elucidate and emphasize the relationships of the various parts, but also to indicate their deeper and less obvious import. Then, too, the pointing out of unsolved problems and incomplete features, whether of detail or theory, will at least be of service in stimulating the constructive imagination of the reader, even if it does not lead to additional contributions in these particular lines.

The first group of lectures deals with the theory of systems of curves on an algebraic surface, with especial reference to properties that are invariant under birational transformations and the kinds of surfaces that admit given systems. This is followed by an account of Picard's theory of linear exact differentials of the first kind on an algebraic surface, a subject of great interest not only from the geometric point of view but also on account of its connection with the slowly developing theory of functions of two variables.

The second set of lectures begins with a clear and simple account of the Riemann hypotheses relative to a restricted portion of space. By the introduction of additional hypotheses the geometry is extended to an unlimited space. With restriction to space of constant curvature three essentially distinct types of geometry are considered, the spherical and elliptic, the lobachevskian, and the euclidean, — these corresponding to positive, negative, and zero curvature respectively. In the first two cases the non-euclidean spaces investigated by Clifford, Klein, and Killing are considered in detail.

The last group of lectures occupies considerably the greater half of the book and thus is afforded a correspondingly larger scope for breadth and variety of treatment, which has been utilized by the author in a most thorough and scholarly manner. The subject, too, is of peculiar interest, since until recent years the divergent series has been looked upon by mathematicians as a monstrosity to be carefully excluded from the well-regulated society of exact science. As, however, the solutions of many differential equations present themselves in the form of divergent series, this subject has during the past twenty years made good its claims to serious recognition, and, thanks to the labors of Poincaré, Stieltjes, Borel, and others, a substantial foundation has been laid for a theory of divergent series. To determine the conditions under which a divergent series may be manipulated as the analytic representative of an unknown function, to develop the properties of the function, and to formulate methods for deriving a function uniquely from the series constitute the chief aim of the theory and form the subject matter of the lectures. The greater share of discussion is given to the last named object. The attention is here directed towards convergent series of rational functions, and the valuable instrumentality of algebraic continued fractions leading to a somewhat full and independent discussion of the theory of such fractions.

A few typographical errors have been observed, none of which would be confusing to the reader. The formulas in x, pages 33–34, should, of course, be expressed in terms of z. The word *sixth*, page 29, line 20, is apparently incorrect, as Humbert states explicitly (*Liouville*, 1893, page 436) that the minimum degree of hyperelliptic surfaces is not yet determined, but he believes it to be *eight*, and considers a number of cases of surfaces of that degree (pages 436–449).

The book is unfortunately printed on very thick paper; while not bulky, it could have been made into a more tasty and compact volume of less than half the thickness.

J. I. HUTCHINSON.

CORNELL UNIVERSITY.

CORRECTION.

The following correction should be made in the paper by Mr. Lennes in the October BULLETIN: Page 14, lines 14–16, *for* where M is the difference $\cdots$ of $f(x)$ on ab *read* where M is twice the least upper bound of the absolute value of $f(x)$ on ab.

NOTES.

BY arrangement with The Macmillan Company a limited number of copies of the Chicago Congress Mathematical Papers are offered to members of the AMERICAN MATHEMATICAL SOCIETY at $1.50 per volume, one half the former price. Orders should be sent direct to the Macmillan Company or to the Secretary of the Society, no commission being allowed to agents. Details concerning the book will be found in the advertising pages of the BULLETIN.

THE seventy-sixth annual meeting of the British association for the advancement of science was held at York, England, August 3 to 8. Professor R. LANKESTER was president of the association, and Dr. E. H. GRIFFITHS president of section A, mathematics and physics. The papers of mathematical interest were: "On the notation and use of vectors," by Professor O. Henrici; "On the irregular motion of the earth's pole," by Mr. E. H. Hills and Professor J. Larmor; "Expansions in products of oscillatory functions," by Professor A. C. Dixon; "On the motion of a particle in a cyclone," by Professor W. H. H. Hudson; "On hypereven numbers," by Lt. Col. A. Cunningham; "On the Lagrange equation," by Professor A. R. Forsyth; "Two new symmetric functions," by Major P. A. MacMahon; "Finite groups," by Mr. H. Hilton; "On multiple series," by Professor T. J. I' A. Bromwich; "Many valued functions of real variables," by Mr. A. R. Richardson; "On methods of computing Bessel functions for large values of the argument," by Professor A. Lodge. The next meeting is to be held at Leicester, that of 1908 at Dublin, and in 1909 the association will for the third time visit Canada, the meeting to be held in Winnipeg.

THE Royal Belgian academy of sciences announces the following prize problems in astronomy and mathematics for the year 1907 (each prize is 800 francs):

(1) To find, in altitude and azimuth, the expressions of the principal terms of the periodic deviations of the vertical, under the hypothesis of the non-coincidence of the centers of gravity of the crust and core of the earth.

(2) Between the elements of two forms of the second order (two non-superposed plane systems, or a plane and a bundle, or two bundles with different vertices) a quadratic correspondence is established (in the sense of Reye, Geometrie der Lage, volume 2, chapter 22). It is desired to study the system of elements obtained by joining or by intersection of pairs of homologous elements in the two forms of the second order.

THE foreign universities below offer courses in mathematics during the year 1906–1907 as follows:

CAMBRIDGE UNIVERSITY. — *Michaelmas term*, 1906. — By Professor A. R. FORSYTH: Abel's theorem and abelian functions, three hours. — By Professor Sir G. H. DARWIN: Theory of potential and attractions, three hours. — By Professor Sir R. S. BALL: Spherical astronomy, three hours. — By Professor J. LARMOR: Electricity and magnetism, three hours. — By Professor B. HOPKINSON: Applied mathematics, two hours. — By

Dr. E. W. Hobson: Representation of functions by series, three hours.—By Dr. H. F. Baker: Theory of functions, three hours; By Mr. H. W. Richmond: Analytic geometry, three hours.—By Dr. A. N. Whitehead: Principles of mathematics, three hours.—By Mr. A. Berry: Elliptic functions, Bessel functions and Fourier series, three hours.—By Mr. A. Munro: Hydrodynamics and sound, I, three hours.—By Mr. J. H. Grace: Invariants and geometric applications, three hours.—By Mr. E. W. Barnes: Taylor's series, three hours.

Lent term, 1907.—By Professor A. R. Forsyth: Calculus of variations, three hours.—By Professor Sir G. H. Darwin: Figure of the earth and precession, three hours.—By Professor Sir R. S. Ball: Spherical astronomy, II, three hours.—By Professor J. Larmor: Electrodynamics, three hours.—By Professor B. Hopkinson: Applied mathematics, two hours.—By Dr. E. W. Hobson: Vibrations and sound, three hours.—By Dr. H. F. Baker: Theory of continuous groups, three hours.—By Mr. H. W. Richmond: Analytic geometry, II, three hours.—By Mr. A. N. Whitehead: Principles of mathematics, II, three hours.—By Mr. B. A. Herman: Hydromechanics, I, three hours; Hydrodynamics, three hours.—By Mr. C. T. Bennett: Line geometry, three hours.—By Mr. E. W. Barnes: Linear differential equations, three hours.

Easter term, 1907.—By Professor J. Larmor: Theory of gases and thermodynamics, two hours.—By Mr. H. W. Richmond: Projective geometry, three hours.—By Mr. A. N. Whitehead: Noneuclidean geometry, three hours.—By Mr. A. Young: Theory of invariants, three hours.—By Mr. G. H. Hardy: Integral functions, three hours.—By Mr. R. R. Webb: Definite integrals, three hours.

University of Berlin (winter semester.)—By Professor H. A. Schwarz: Differential calculus, four hours; synthetic geometry, four hours; Geometric conics, two hours; Seminar, two hours; Colloquium, two hours.—By Professor G. Frobenius: Algebra, four hours; Seminar, two hours.—By Professor F. Schottky: Integral calculus; four hours; General theory of functions, four hours; Seminar, two hours.—By Professor G. Hettner: Definite integrals, two hours.—By Professor J. Knoblauch: Determinants, four hours; Curved surfaces, four hours; Twisted curves, one hour.—By Professor R. LEHMANN-FILHÉS: Analytic geometry, four hours. By Professor E. LANDAU: Theory of numbers, four hours. — By Dr. I. SCHUR: Ordinary differential equations, four hours.

UNIVERSITY OF BONN (winter semester.) — By Professor E. STUDY: Non-euclidean geometry, four hours; Introduction to analytic mechanics, four hours; Seminar, two hours. — By Professor G. KOWALEWSKI: Calculus, II, four hours; Fourier's series, two hours; Geometry of numbers, two hours; Seminar, two hours. — By Professor F. LONDON: Analytic geometry of two and three dimensions, four hours; Descriptive geometry with exercises, three hours; Seminar, two hours. — By Dr. E. SCHMIDT: Introduction to algebra, three hours; Determinants, two hours.

UNIVERSITY OF BOLOGNA. — By Professor C. ARZELÀ: Elements of differential geometry, differential equations and calculus of variations, three hours. — By Professor E. CESÀRO: Mathematical theory of elasticity, three hours. — By Professor L. DONATI: General methods of mathematical physics, Lorentz's theory of electrons, three hours. — By Professor L. PINCHERLE: Elements of algebraic functions and integrals, three hours; Continuous groups of transformations, three hours.

UNIVERSITY OF CATANIA. — By Professor G. LAURICELLA: Functional equations of potential and electrostatics, four and a half hours. — By Professor G. PENNACCHIETTI: Dynamics of solid bodies, II, four and a half hours. — By Professor M. PIERI: Selected chapters of differential geometry, three hours. — By Professor C. SEVERINI: Groups of substitutions and the Galois theory of algebraic equations, three hours.

UNIVERSITY OF GENOA. — By Professor G. FUBINI: Differential and integral equations, developments in series of normal functions, three hours. — By Professor G. LORIA: General theory of algebraic curves and surfaces, three hours. — By Professor O. TEDONE: Vector analysis, kinematics and statics of continua, three hours.

UNIVERSITY OF MESSINA. — By Professor G. BAGNERA: Algebraic functions of two variables, and their integrals, three hours. — By Professor R. MARCOLONGO: Electrodynamics, Lorentz's theory and its applications to optics, three hours. — By Professor V. MARTINETTI: General theory of plane curves, cubic curves, three hours. — By Professor S. ORLANDO: Partial differential equations of mathematical physics, three hours. — By Professor G. VIVANTI: Applications of the theory of contact transformations, three hours.

UNIVERSITY OF NAPLES. — By Professor F. AMODEO: History of mathematics from 1200 to 1800, four and a half hours. — By Professor A. CAPELLI: Theory of groups and applications to analysis, four and a half hours. — By Professor P. DEL PEZZO: The non-euclidean geometries, four and a half hours. — By Professor A. DEL RE: Algebra of logic, four and a half hours. — By Professor D. MONTESANO: Hypergeometry, four and a half hours; Imaginary elements of geometry, four and a half hours. — By Professor L. PINTO: Electrostatics, three hours. — By Professor F. SIACCI: General theory of dynamical equations, three hours.

UNIVERSITY OF PADUA. — By Professor F. D'ARCAIS: Theory of analytic functions; assemblages and applications, four and a half hours. — By Professor A. FAVARO: History of mathematics in Italy to the sixteenth century, three hours. — By Professor P. GAZZANIGA: Theory of numbers, three hours. — By Professor T. LEVI-CIVITA: Electromagnetic fields, dynamics of electrons, four and a half hours. — By Professor A. PADOA: Mathematical logic, one hour. — By Professor G. RICCI: Mathematical physics, elasticity and application to optics, four hours. — By Professor F. SEVERI: Theory of algebraic functions of one and two variables with geometric applications, four hours. — By Professor G. VERONESE: Geometry of hyperspace, four hours.

UNIVERSITY OF PALERMO. — By Professor F. GERBALDI: Theory of functions and applications, three hours. — By Professor G. B. GUCCIA: Theory of plane curves and surfaces, four and a half hours. — By Professor G. TORELLI: Partial differential equations of mathematical physics, four and a half hours. — By Professor A. VENTURI: Theory of the form of the earth, theory of the tides, four and a half hours.

UNIVERSITY OF PAVIA. — By Professor E. ALMANSI: Mathematical theory of elasticity, three hours. — By Professor F. ASCHIERI: Projective properties of algebraic forms in hyperspace, with applications, three hours. — By Professor L. BERZOLARI: General theory of functions, elliptic functions, three hours. — By Professor E. PASCAL: Theory of transformations of elliptic functions, with application to the quintic equation; Differential forms of order higher than two, three hours.

UNIVERSITY OF PISA. — By Professor E. BERTINI: Foundations of the geometry of hyperspaces, geometry of general algebraic curves, two hours. — By Professor L. BIANCHI: Theory of functions and elliptic functions, four and a half hours. — By Professor U. DINI: Linear differential equations, four and a half hours. — By Professor A. MAGGI: Principles of electricity, four and a half hours. — By Professor P. PIZZETTI: General notions of spherical astronomy and planetary perturbations, three hours.

UNIVERSITY OF ROME. — By Professor G. BISCONCINI: Mathematical theory of elasticity, with applications, three hours. — By Professor G. CASTELNUOVO: Cremona transformations and linear systems of curves, three hours. — By Professor V. CERRUTI: Calculus of variations, three hours.

UNIVERSITY OF TURIN. — By Professor G. MORERA: Hydrodynamics, three hours. — By Professor E. D'OVIDIO: Theory of functions, Jacobi's and Weierstrass's theories of elliptic functions, three hours. — By Professor G. PEANO: Mathematical logic, two hours. — By Professor C. SEGRE: Groups in geometry, three hours. — By Professor C. SOMIGLIANA: Mathematical theory of elasticity and its applications, three hours.

PROFESSOR E. CESÀRO, of the University of Naples, has been appointed professor of analytic mechanics at the University of Bologna.

At the University of Naples, Professor P. DEL PEZZO (higher and differential geometry) and Professor D. MONTESANO (projective and synthetic geometry) have exchanged chairs.

PROFESSORS L. BIANCHI and P. PIZZETTI, of the University of Pisa, and Professor C. SEGRE, of the University of Turin, have been elected members of the Royal institute of Venice.

PROFESSOR F. ENRIQUES, of the University of Bologna, has been elected corresponding member of the Royal academy of the Lincei. Professor P. PAINLEVÉ, of the University of Paris, has been elected foreign member.

THE University of Aberdeen has conferred the honorary degree of doctor of science on Professor G. VERONESE, of the University of Padua.

DR. M. ADLER has been appointed docent in descriptive geometry at the technical school of Vienna, and Dr. K. BOPP to a similar position at the University of Heidelberg.

PROFESSOR P. STÄCKEL, of the technical school at Hanover, has been elected corresponding member of the Royal society of Göttingen.

ON account of his duties connected with the reorganization of the technical school at Turin, Professor V. VOLTERRA, of the University of Rome, has been relieved from lecturing during the present year.

PROFESSOR A. SOMMERFELD, of the technical school at Aachen, has been appointed professor of theoretical physics at the University of Munich.

PROFESSOR G. B. HALSTED, recently of Kenyon College, Ohio, has been appointed professor of mathematics at the Colorado State normal school, Greeley, Colorado.

AT the University of Missouri, Dr. L. D. AMES has been promoted to an assistant professorship of mathematics.

DR. H. A. CONVERSE, recently of the Baltimore Polytechnic Institute, has been appointed professor of mathematics in Davis and Elkins College, Elkins, W. Va.

MR. C. G. SIMPSON has been appointed instructor in mathematics in the State University of Iowa.

DR. C. H. SISAM, of the U. S. Naval Academy, has been appointed instructor in mathematics at the University of Illinois.

MISS M. H. WALBRIDGE has been appointed professor of physics and mathematics at Wells College, Aurora, New York.

DR. A. L. P. WERNICKE, formerly professor of modern languages in the State College of Kentucky, has been appointed instructor in mathematics in Washington University, St. Louis.

PROFESSOR LUDWIG BOLTZMANN, professor of theoretical physics at the University of Vienna, died by his own hand, September 5, at the age of 62 years.

Catalogues of second-hand mathematical works: Alfred Lorenz, Leipzig, Kurprinzstrasse 10, Katalog 165, logic (logic of mathematics; arithmetic, geometric and infinitesimal methods etc.).—Franz Pietzker, Tübingen, Verzeichnis No. 366, mathematics, physics and astronomy.—Galloway and Porter, Cambridge, England, catalogue No. 30, containing 466 titles in mathematics.—Max Weg, Leipzig, Leplaystrasse 1, catalogue No. 103, geodesy, astronomy, mathematics and physics, 2682 titles.—Gustav Fock, Leipzig, Schlossgasse 7–9, Antiquariats-Verzeichnis, No. 288, 4400 titles in mathematics and physics, including the library of the late Professor O. Stolz.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ANDRADE (J.). Les fonctions angulaires dans la géométrie de l'ajustage. 8vo. (*L'enseignement mathématique* 8, pp. 257–281.)

BARTOLI (E.). See FREYCINET (C. DE).

BERZOLARI (L.). See ENCYKLOPÄDIE.

BRIOSCHI (F.). Opere matematiche, pubblicate per cura del comitato per le onoranze a Francesco Brioschi. Vol. IV. Milano, Hoepli, 1906. 4to. 8 + 418 pp.

ENCYKLOPÄDIE der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Herausgegeben im Auftrage der Akademien der Wissenschaften zu Göttingen, Leipzig, München und Wien sowie unter Mitwirkung zahlreicher Fachgenossen. (In 7 Bänden.) Vol. III: Geometrie, redigiert von W. F. Meyer. Teil 2, Heft 3: H. G. Zeuthen, Abzahlende Methoden; L. Berzolari, Allgemeine Theorie der höheren ebenen algebraischen Kurven. Leipzig, Teubner, 1906. 8vo. Pp. 257–455.

FREYCINET (C. DE). Saggio sulla filosofia delle scienze. Analisi; meccanica. Traduzione da E. Bartoli. Bari, 1906. 12mo. 330 pp. M. 3.00

GRÜNFELD (E.). Zur Theorie der linearen Differentialgleichungen. (Progr.) Wien, 1906. 8vo. 20 pp.

GRUNER (P.). Tabellen für die Exponentialfunktion mit negativen Exponenten. Leipzig, Hirzel, 1906. 8vo. 15 pp. M. 1.00

HIESSL (M.). Einige Elemente der vierdimensionalen Geometrie. (Progr.) Stockerau, 1906. 8vo. 22 pp.

HOLTAPPEL (W. T.). See SCHOUTEN (G.).

MEYER (W. F.). See ENCYKLOPÄDIE.

MIRINNY (L.). Isonolyse (résolution générale des équations). Méthode primordiale. Pentisonolyse (résolution générale de l'équation du cinquième degré). (I. Algébrique proprement dite ou par radicaux; II. Exponentielle; III. Trigonométrique; IV. Géométrique; V. Troponomique.) Aperçu sommaire initial. Edition de 1896 élucidée et abrégée. Paris, Marquet, 1906. 16mo. 4 pp., 3 plates.

MOSNAT (E.). Problèmes de géométrie analytique. Vol. I: A l'usage des candidats aux écoles (centrale, ponts et chaussées, mines, polytechnique, normale) et des aspirantes à l'agrégation des jeunes filles. 3e édition. Paris, Vuibert et Nony [1906]. 8vo. 528 pp.

Picard (E.) et Simart (G.). Théorie des fonctions algébriques de deux variables indépendantes. Vol. II. Paris, Gauthier-Villars, 1906. 8vo. 6 + 528 pp.

Schouten (G.). Differentiaalrekening. Bewerkt naar het college van Schouten door W. J. Holtappel. 1. studiejaar. Delft, 1906. 8vo. M. 1.50

Simart (G.). See Picard (E.).

Vivanti (G.). Elementi della teoria delle funzioni poliedriche e modulari. Milano, 1906. 12mo. 8 + 428 pp. M. 2.50

Wirth, (J.). Ueber die Elementarteiler einer homogenen Substitution. (Diss.) Freiburg, 1906. 8vo. 31 pp.

Wirtinger (W.). Ueber die Entwicklung einiger mathematischer Begriffe in neuerer Zeit. Wien, Hölder, 1906. 8vo. 20 pp. M. 0.60

Young (W. H.) and Young (G. C.). Theory of sets of points. Cambridge, University Press, 1906. 8vo. 328 pp. Cloth. 12 s.

Zeuthen (H. G.). See Encyklopädie.

II. ELEMENTARY MATHEMATICS.

Adler (A.). Theorie der geometrischen Konstruktionen. Leipzig, Göschen, 1906. 12mo. 8 + 290 pp. Cloth. (Sammlung Schubert, No. LII.) M. 9.00

——. Zur Konstruktion des regelmassigen Siebzehneckes. (Progr.) Wien, 1906. 8vo. 20 pp.

Auflösungen für die Trigonometrie. Für Schule und Praxis herausgegeben von einem Fachmann. Neustrelitz, Heydmann & Kett, 1906. 8vo. 13 pp. M. 0.60

Boucheny (G.). See Perrin (E.).

Braune (A.). Raumlehre für Volks-, Bürger- und Fortbildungsschulen, sowie für Präparandenanstalten. Nach methodischen Grundsätzen bearbeitet. Bearbeitet von F. Skorczyk. 8te, verbesserte Auflage. Halle, Schroedel, 1906. 8vo. 5 + 96 pp. Boards. M. 0.90

Cambier (J.) et Manceau (R.). Le calcul pratique, 4000 exercices gradués pour les cours. Livret des vétérans. Paris, Brodard, 1906. 16mo. 48 pp. Fr. 0.35

Clive's mathematical tables, containing twenty-two tables of logarithms, antilogarithms, natural and logarithmic trigonometrical functions, and circular measure. Specially designed for rapid and accurate calculation. London, Clive, 1906. 8vo. 58 pp. Cloth. (University tutorial series.) 1s. 6d.

Cracknell (A. G.). See Workman (W. P.).

Ehrig (G.). Geometrie für die Zwecke des praktischen Lebens. Zweiter Teil: Geometrie des Raumes. Leipzig, Leineweber, 1906. 8vo. 8 + 120 pp. Cloth. M. 2.25

——. Geometrie für Baugewerkenschulen und verwandte technische und gewerbliche Lehranstalten mit besonderer Berücksichtigung der praktischen Anwendungen. Teil II: Geometrie des Raumes. Leipzig, Leineweber, 1906. 8vo. 8 + 120 pp. Cloth. M. 2.25

French (C. H.) and Osborn (G.). Matriculation graphs. London, Clive, 1906. 12mo. 70 pp. Limp leather. (University tutorial series.) 1s.

FUSS (K.). Lehrbuch der Buchstabenrechnung und Algebra für den Schul- und Selbstunterricht. 5te, verbesserte und vermehrte Auflage. Nürnberg, Korn, 1906. 8vo. 11 + 317 pp. M. 3.00

HAAS (A.). Lehrbuch uber den binomischen und polynomischen Lehrsatz, die arithmetischen Reihen hoherer Ordnung und die unendlichen Reihen, mit 295 Fragen und Antworten, 202 Erklärungen, 502 meist gelösten Aufgaben und einem Formelnverzeichnis, zum Selbststudium und dem Gebrauch an Schulen bearbeitet nach dem System Kleyer. Bremerhaven, Vangerow, 1906. 8vo. 7 + 370 pp. M. 8.00

HUE (T.) et VERNADET (A.). Géométrie (1re année). Paris, Delagrave [1906]. 8vo. 511 pp. (Bibliothèque des écoles normales.)

J. (F.). Eléments de trigonométrie rectiligne, avec de nombreux exercices. Paris, Poussielgue [1906]. 16mo. 7 + 257 pp. (Cours de mathématiques élémentaires.)

KÖSTLER (H.). Leitfaden der ebenen Geometrie für höhere Lehranstalten, neu herausgegeben von A. Holtze. Heft 1: Kongruenz. 6te, teilweise umgearbeitete Auflage. Halle, Nebert, 1906. 8vo. 86 pp. Boards. M. 1.35

——. Heft 3: Aehnlichkeitslehre. 4te, vielfach umgearbeitete Auflage. Halle, Nebert, 1906. 8vo. 108 pp. Boards. M. 1.50

KUTNEWSKY (M.). See MÜLLER (H.).

LOUIS (A.). Quelques mots sur l'analyse des problèmes d'arithmétique. Le Mans, 1906. 8vo. 71 pp. M. 1.20

MACÉ DE LÉPINAY (A.). Compléments d'algèbre et notions de géométrie analytique à l'usage des élèves se destinant aux mathématiques spéciales et des jeunes filles visant à l'agrégation. 5e édition. Paris, Vuibert et Nony [1906]. 8vo. 486 pp.

MANCEAU (R.). See CAMBIER (J.).

MATRICULATION modern answers. Mathematics, from September 1902 to June 1906. London, Clive, 1906. 12mo. 156 pp. Cloth. 2s.

MÜLLER (H.). Mathematisches Unterrichtswerk. Abteilung I: Die Mathematik auf den Gymnasien und Realschulen. In Verbindung mit dem Verfasser für bayerische Lehranstalten herausgegeben von M. Zwerger. Teil 1: Lehraufgabe der 5. und 6. Gymnasial-, bez. der 3. und 4. Realschulklasse. Leipzig, Teubner, 1906. 8vo. 6 + 138 pp. Boards. M. 1.60

——. Abteilung 1, Teil 2: Lehraufgabe der 7. und 8. Gymnasial-, bez. der 5. und 6. Realschulklasse. Leipzig, Teubner, 1906. 8vo. 6 + 162 pp. Boards. M. 2.00

——. Abteilung II: Sammlung von Aufgaben aus der Arithmetik, Trigonometrie und Stereometrie. Im Anschluss an die Teile A I und A II der Müller und Kutnewskyschen Aufgabensammlung und in Verbindung mit den Verfassern für bayerische Lehranstalten herausgegeben von M. Zwerger. Leipzig, Teubner, 1906. 8vo. 8 + 276 pp. Boards. M. 2.60

MÜLLER (H.) und KUTNEWSKY (M.). Sammlung von Aufgaben aus der Arithmetik, Trigonometrie und Stereometrie. Ausgabe A: für Gymnasien und Progymnasien. Teil 1. 4te Auflage. Leipzig, Teubner, 1906. 8vo. 7 + 237 pp. Boards. M. 2.20

——. Ausgabe B: für reale Anstalten und Reformschulen. Teil 1. 4te Auflage. Leipzig, Teubner, 1906. 8vo. 8 + 301 pp. Boards. M. 2.80

Osborn (G.). See French (C. H.).

Perrin (E.) et Bouchény (G.). Trigonométrie. Première C et D, mathématiques A et B; écoles navale et de Saint-Cyr; institut agronomique. Paris, Paulin, 1906. 16mo. 8 + 468 pp.

Pionchon (J.). Principes et formules de trigonométrie rectiligne et sphérique. Grenoble, 1906. 8vo. 146 pp. Fr. 5.00

Skorczyk (F.). See Braune (A.).

Toffolotti (C.). Lezioni di aritmetica e algebra, per la terza classe liceale. Venezia, Visentini, 1906. 16mo. 103 pp. L. 1.50

Traub (K.). Elementare Berechnung der Seiten der regulären Vierunddreissig- und Siebzehnecke. Karlsruhe, 1906. 8vo. 23 pp., 1 plate. M. 0.60

Vernadet (A.). See Hue (T.).

Workman (W. P.) and Cracknell (A. G.). Matriculation geometry, being sections I–IV of "Geometry, theoretical and practical." London, Clive, 1906. 12mo. 448 pp. Cloth. (University tutorial series.) 3s. 6d.

Zwerger (M.). See Müller (H.).

III. APPLIED MATHEMATICS.

Aranaz (R.). Perspectiva. Lecciones elementales. 3a edición. Granada, Traveset, 1905. 8vo. 45 pp., 3 plates. P. 3.00

Badon Ghijben (J.). Gronden der beschrijvende meetkunde. Bewerkt door N. G. Grotendorst en J. W. C. Beelenkamp. 9te druk. Deel I. Breda, 1906. 8vo. 19 + 175 pp., 51 plates. M. 6.00

Beelenkamp (J. W. C.). See Badon Ghijben (J.).

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THE SEPTEMBER MEETING OF THE SAN FRANCISCO SECTION.

THE tenth regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of California on Saturday, September 29, 1906. The following sixteen members were present:

Professor R. E. Allardice, Professor H. F. Blichfeldt, Mr. A. J. Champreux, Professor G. C. Edwards, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Professor A. O. Leuschner, Dr. J. H. McDonald, Dr. W. A. Manning, Professor H. C. Moreno, Professor C. A. Noble, Dr. T. M. Putnam, Professor Irving Stringham, Mr. L. C. Walker, Professor E. J. Wilczynski.

Professor R. E. Allardice occupied the chair at both the morning and afternoon sessions. The following officers were elected for the ensuing year: Professor E. J. Wilczynski, chairman; Dr. W. A. Manning, secretary; Professor M. W. Haskell, Professor H. F. Blichfeldt, and Dr. W. A. Manning, program committee. It was arranged to hold the next meeting on Saturday, February 23, at Stanford University.

The following papers were read at this meeting:

(1) PROFESSOR R. E. ALLARDICE: "Additional note on the multiple points of unicursal curves."

(2) PROFESSOR H. F. BLICHFELDT: "A theorem concerning the Sylow subgroups of simple groups."

(3) PROFESSOR M. W. HASKELL: "On the collineation group belonging to triangles quadruply in perspective."

(4) PROFESSOR L. M. HOSKINS: "The effect of viscosity on the propagation of free vibrations in an elastic solid."

(5) DR. W. A. MANNING: "On the order of primitive groups."

(6) PROFESSOR R. E. MORITZ: "On the symbolic representation of quotiential coefficients of the second order."

The paper by Professor Moritz was read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. If a unicursal curve be given by the equations $x = (a_0, a_1, \cdots, a_n)(\lambda, 1)^n$, etc., the multiple points depend on the

quantities $A = (a_0, b_1, c_2)$, etc. It is known that the multiple point of a cubic curve is given by the hessian of the cubic $(A, B, C, D)(\lambda, 1)^3$. In the case of curves of a higher order the problem becomes complicated on account of the fact that the quantities (a_0, b_1, c_2), (a_0, b_1, c_3), etc., are not independent. In a previous paper Professor Allardice gave a theorem for the quartic curve analogous to the theorem for the cubic stated above; and in an additional note, read at this meeting, curves of the fifth degree in space of two, three and four dimensions were considered.

2. Frobenius has proved ("Ueber auflösbare Gruppen V," *Sitzungsberichte der Berliner Akademie*, 1901, page 1324) that in a simple group G of composite order there must be contained a subgroup H whose order is a power of a prime p, and a substitution S of order prime to p which transforms H into itself yet is not commutative with every substitution of H. Professor Blichfeldt proves that we may use for H a subgroup of G of order p^λ, this being the highest power of p which divides the order of G, provided such a subgroup is abelian, or of some other specified type.

3. Professor Haskell showed that it is possible to construct a system of four triangles, each pair quadruply in perspective and inscribed in a conic, and that the perspective reflections thus defined generate a group of 72 collineations which is equivalent to the group

$$y_1 : y_2 : y_3 = \gamma^\lambda x_p : \gamma^\mu x_q : \gamma^\nu x_r,$$

where $\gamma^6 = 1$, $\lambda + \mu + \nu \equiv 0 \pmod{6}$ and p, q, r are 1, 2, 3 in any order. This group might well be regarded as trivial, apart from its relation to the above configuration. This configuration exhibits further an interesting relation to the nine points of inflexion of a cubic and their harmonic polars.

4. In an isotropic elastic solid without internal friction or viscosity, dilatational and rotational vibrations travel with velocities which depend upon the elastic moduli and the density but are independent of the frequency. The object of Professor Hoskins's paper is to investigate how this result is affected by viscosity. The discussion is restricted to the case in which viscosity is specified by a single modulus ν, the differential

equations for the displacements being obtained from those applying to a non-viscous solid by substituting $n + \nu d/dt$ for the rigidity-modulus n. It is found that the velocity of propagation is not independent of the frequency, and that for each kind of vibration it may have any value from 0 up to the value corresponding to no viscosity. It thus appears that viscosity furnishes a sufficient explanation of the fact that earthquake vibrations lasting but a fraction of a minute near their place of origin are recorded for several hours at distant places; the explanation being that the original complex disturbance analyzes into simple vibrations which travel with velocities varying from the greatest value (corresponding to no viscosity) down to zero.

5. Dr. Manning presented the following theorem: Let q be one of the numbers 2, 3, 4; p any prime greater than $q + 1$; the degree of a primitive group which contains a substitution of order p with q cycles (without containing the alternating group) cannot exceed $pq + q$. This is directly connected with the similar theorem stated, in part without proof, by Jordan in the *Bulletin de la Société mathématique de France*, volume 1, page 221.

6. Professor Moritz's paper deals with analogies between certain differentiation formulas and the corresponding quotientiation formulas. It is first shown that for a function of n dependent variables $y = F(u_1, u_2, \cdots, u_n)$

$$\frac{qy}{qx} = \left(\frac{qy}{qu_1}\right)\frac{qu_1}{qx} + \left(\frac{qy}{qu_2}\right)\frac{qu_2}{qx} + \cdots + \left(\frac{qy}{qu_n}\right)\frac{qu_n}{qx},$$

where the parentheses are used in the eulerian sense to denote partial variations of the variables. The second quotiential coefficient is given by

$$\frac{q^2y}{qx^2}\ \frac{qy}{qx} = \sum_{i=1}^{n}\left(\frac{qy}{qu_i}\right)\left(\frac{q^2y}{qu_i^2}\right)\frac{\overline{qu_i}^2}{qx} + 2\sum_{\substack{k=2\\ i<k}}^{k=n}\left(\frac{qy}{qu_k}\right)\left(\frac{q^2y}{qu_i qu_k}\right)\frac{qu_i}{qx}\,\frac{qu_k}{qx}$$
$$+ \sum_{i=1}^{n}\left(\frac{qy}{qu_i}\right)\frac{qu_i}{qx}\,\frac{q^2u_i}{qx^2}.$$

It is pointed out that this expression may be written symbolically thus

$$\frac{\partial^2 y}{\partial x^2}\frac{\partial y}{\partial x}=\left[\sum_{i=1}^{n}\left(\frac{\partial y}{\partial u_i}\right)\frac{\partial u_i}{\partial x}\right]^{(2)}+\sum_{i=1}^{n}\left(\frac{\partial y}{\partial u_i}\right)\frac{\partial u_i}{\partial x}\frac{\partial^2 u_i}{\partial x^2},$$

where the exponent in parenthesis signifies that the expression to which it is attached is to be squared, and after squaring

$$\left(\frac{\partial y}{\partial u}\right)^2,\quad\left(\frac{\partial y}{\partial u}\right)\left(\frac{\partial y}{\partial v}\right),\quad\left(\frac{\partial y}{\partial v}\right)^2$$

are to be replaced by

$$\left(\frac{\partial y}{\partial u}\right)\left(\frac{\partial^2 y}{\partial u^2}\right),\quad\left(\frac{\partial y}{\partial v}\right)\left(\frac{\partial^2 y}{\partial u\partial v}\right),\quad\left(\frac{\partial y}{\partial v}\right)\left(\frac{\partial^2 y}{\partial v^2}\right),$$

respectively.

W. A. MANNING,
Secretary of the Section.

PROJECTIVE DIFFERENTIAL GEOMETRY.

AN ABSTRACT OF FOUR LECTURES DELIVERED AT THE NEW HAVEN COLLOQUIUM, SEPTEMBER 5–8, 1906.

BY PROFESSOR E. J. WILCZYNSKI.

THESE four lectures were devoted to an exposition of the principal results belonging to the subject of projective differential geometry. The place of this subject in a systematic treatment of geometry is indicated by the following discussion.

A first important basis for the classification of the various geometries is furnished by the group concept. There is metric geometry, projective geometry, the geometry of the birational transformations, to mention only the most important. Together with this classification by means of the characteristic groups, there is the distinction between differential and integral geometry. The differential properties of a geometric configuration merely depend upon the fact that in a certain, perhaps very small, region, certain conditions of continuity are satisfied, that derivatives of a certain order exist, etc. These differential properties are studied by means of the differential calculus and do not depend upon the nature of the configuration (curve, surface, etc.) as a whole, but merely upon its character in a certain neighborhood. Opposed to the differential, we have the integral properties which depend upon the configuration as a whole. Thus within each geometry, characterized by its group we may still distinguish two divisions; we have differential and integral metric, differential and integral projective geometry; the geometry of birational transformations is likewise composed of two such divisions. Integral projective and differential metric geometry have been systematically developed. It is noticeable that the same is not true of differential projective or of integral metric geometry. In the nature of things differential geometry is more easily accessible than integral geometry. For the relation between the two is precisely that of the differential to the integral calculus. From the properties of differential geometry those of integral geometry may be obtained by integration, a process which requires the invention of special methods for every particular case. If, therefore, we can speak of an integral projective geometry as existing, it is nevertheless only in a very special sense. In fact, the configurations of projective geometry have, in almost all cases, been assumed to be algebraic, a restriction which is equivalent to an a priori integration. Leaving the algebraic cases aside, it is clear, therefore, that integral geometry, taken in the general sense, must be preceded by differential geometry. The metric half of this latter subject has been occupying the attention of mathematicians since the days of Monge and Gauss, and in the hands of their successors has reached a high degree of perfection. The same cannot be said of projective differential geometry. These lectures give an account of the first systematic treatment of a part of this subject, namely the projective differential geometry of curves and ruled surfaces.

The theory of curves in a space of $n-1$ dimensions is based upon the theory of the invariants and covariants of a linear homogeneous differential equation of the nth order, a point of view due to Halphen and developed by him in an admirable series of papers. The theory of ruled surfaces is based upon the consideration of a system of linear differential equations of the form

$$(A)\qquad \begin{aligned} y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z &= 0,\\ z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z &= 0. \end{aligned}$$

In connection with this system, consider the infinite group G of transformations

$$(G) \qquad \eta = \alpha(x)y + \beta(x)z, \quad \zeta = \gamma(x)y + \delta(x)z, \quad \xi = f(x),$$

where α, β, γ, δ, f are arbitrary functions of x. The transformations of G convert (A) into another system of the same kind. The problem presents itself: to find the invariants and covariants of the system (A) under the transformations of the group G. This problem has been completely solved.

Let (y_i, z_i), $(i = 1, 2, 3, 4)$ be four pairs of simultaneous solutions of system (A). Then it may be shown that the most general pair of solutions is given by the equations

$$y = \sum_{i=1}^{4} c_i y_i, \quad z = \sum_{i=1}^{4} c_i z_i,$$

involving four arbitrary constants $c_1, \cdots c_4$, provided that the determinant

$$D = \begin{vmatrix} y_1' & y_2' & y_3' & y_4' \\ z_1' & z_2' & z_3' & z_4' \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

is not identically equal to zero. Four such pairs of functions, whose determinant does not vanish, and which satisfy system (A), constitute a fundamental system of simultaneous solutions. Conversely any four pairs of this kind determine a system of form (A) of which they constitute a fundamental system.

Interpret $(y_1, \cdots, y_4)$ and $(z_1, \cdots, z_4)$ as the homogeneous coordinates of two points p_y and p_z. As the independent variable x changes, p_y and p_z describe two curves C_y and C_z, the integral curves of system (A). There is a definite correspondence between the points of these curves, those being corresponding points which belong to the same value of x. Join corresponding points p_y and p_z by a straight line h_{yz}. The ruled surface S which is the locus of these lines is called the integrating ruled surface of system (A). It is easy to show that any non-developable ruled surface may be defined by a system of form (A); the developables are excluded by the condition that the determinant D shall not be equal to zero.

The transformations of the group G leave the ruled surface S invariant. In fact, their geometric significance is merely to replace the two curves C_y and C_z upon S by two other curves C_η and C_ζ upon S, the point-to-point correspondence being again determined by the generators of S. The transformations of G further involve the most general change of the independent variable. An invariant of system (A) under the transformations of the group G therefore has a significance for the ruled surface S which is independent of the particular curves upon S chosen as fundamental curves, and of the choice of the independent variable. Moreover this significance is of a projective nature, since any projective transformation of the surface S gives rise to the same system of form (A), and since the most general integrating ruled surface of (A) is a projective transformation of any particular one.

Upon these considerations the lecturer based his theory of ruled surfaces which, together with some known results, contains a great variety of new ones. It is impossible to give an intelligible account of the details of this theory without greatly exceeding the space at our disposal. It is expounded in detail in the lecturer's recent treatise.[*] This brief abstract will suffice to indicate the character of the subject under investigation and the nature of the methods employed.

ON LOCI THE COORDINATES OF WHOSE POINTS ARE ABELIAN FUNCTIONS OF THREE PARAMETERS.

BY PROFESSOR J. I. HUTCHINSON.

(Read before the American Mathematical Society, September 3, 1906.)

A particular case of a surface the homogeneous coordinates of whose points are theta functions of three variables u_1, u_2, u_3 (connected by a relation $\vartheta = 0$) is given by Humbert[†] and studied in considerable detail by means of this parametric representation. Such a surface may be regarded as a section of a spread S (or three-dimensional locus) in space of four dimensions the coordinates $x_0, \cdots, x_4$ of whose points are five theta functions $\Theta_\iota(u_1, u_2, u_3)$ of the same order m and characteristic (c). It is the partial, or complete intersection of S by a flat space of three dimensions, a flat.

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[*] E. J. Wilczynski. Projective differential geometry of curves and ruled surfaces. Leipzig, B G. Teubner, 1906.

[†] "Sur une surface du sixième ordre liée aux fonctions abéliennes de genre trois," *Liouville*, 1896, pp. 263–293.

For brevity let the symbol $\omega_k (k = 1, \cdots, 64)$ be used to refer to any one of the 64 systems of half-periods. The number of conditions that Θ_ι vanish of order r_k for the half-period ω_k is in general $\frac{1}{6} r_k(r_k + 1)(r_k + 2)$. The function then contains

$$m^3 - \frac{1}{6}\sum_{k=1}^{64} r_k(r_k + 1)(r_k + 2)$$

linear constants. The cases in which this number reduces to 5 are of particular interest since all such functions are linearly expressible in terms of the x_ι.

The degree n of S is the number of non-congruent solutions of three simultaneous equations of the form $\sum a_\iota x_\iota = 0$, which is

$$n = 6m^3 - \sum r_k^3.$$

Combining this with the relation

$$m^3 - \frac{1}{6}\sum r_k(r_k + 1)(r_k + 2) \geqq 5$$

we have

$$n - 30 \geqq \sum(3r_k^2 + 2r_k).$$

Since the right member is positive, it follows that n is not in general less than 30, and for a given n there are only a finite number of cases to be considered.

If, for example, we take the functions $x_1, \cdots, x_4$ employed by Humbert (l. c., page 270) and assume $x_0 = \vartheta_1\vartheta_2$, we find S to be of degree 36, since the coordinate functions vanish to the first order for 12 half-periods. Corresponding to these half-periods are 12 planes lying completely in S. The section of S by the flat $x_0 = 0$ consists of these 12 planes, each counted twice, and two sextics $\vartheta_1 = 0$, $\vartheta_2 = 0$, the first of which is the one discussed by Humbert. The second can be transformed linearly into the first, since ϑ_1 and ϑ_2 permute by the addition of a suitable half-period.

If the x_ι are all even (or odd), the above relations are to be modified, since to the two different sets of values u_k, $-u_k$ of the parameters corresponds only one point x_1. Denote by $2s_k$, $2t_l + 1$ the order of vanishing at a half-period when that order is even or odd respectively. Then

$$6m^3 - \sum_{k=1}^{\kappa} 8s_\kappa^3 - \sum_{l=\kappa+1}^{64} (2t_l + 1)^3 = 2n$$

and the number of linear constants in the x_1 is

$$\tfrac{1}{2}(m^3+\mu) - \tfrac{1}{6} \sum_{k=1}^{\kappa} s_k(s_k+1)(4s_k-1) - \tfrac{1}{6} \sum_{l=\kappa+1}^{64} t_l(t_l+1)(4t_l+5) \geqq 5$$

in which κ and μ have the following values :

1. $\kappa = 64$, $\mu = 8$; m even, Θ even, characteristic zero ;
2. $\kappa = 0$, $\mu = -8$; m even, Θ odd, characteristic zero ;
3. $\kappa = 32$, $\mu = 0$; m even, Θ odd, characteristic zero ;
4. $\kappa = 36$, $\mu = -1$; m odd, Θ $\left\{\begin{smallmatrix}\text{even}\\ \text{odd}\end{smallmatrix}\right\}$, characteristic $\left\{\begin{smallmatrix}\text{odd}\\ \text{even}\end{smallmatrix}\right\}$;
5. $\kappa = 28$, $\mu = 1$; m odd, Θ $\left\{\begin{smallmatrix}\text{even}\\ \text{odd}\end{smallmatrix}\right\}$, characteristic $\left\{\begin{smallmatrix}\text{even}\\ \text{odd}\end{smallmatrix}\right\}$.

As in the general case, we deduce the condition

$$n + 3\mu + 2 - \tfrac{1}{2}\kappa \leqq \sum_{k=1}^{\kappa} (3s_k^2 - s_k) + \sum_{l=\kappa+1}^{64} (3t_l^2 + 2t_l),$$

which gives a restricted number of choices for m, s_k, t_l when n is given. The least value that n can have appears to be 12. This occurs in the two following cases :

I. Let α, β, γ denote three different systems of half-periods, and (α), (β), (γ) represent their characteristics. These can be taken odd without loss of generality. For, if (c_1), (c_2), (c_3) denote any three characteristics, assume (g) so that $(c_1 + g) = (\alpha)$, $(c_2 + g) = (\beta)$, $(c_3 + g) = (\gamma)$ in which the right members are odd. Then we have the three groups*

$$\begin{aligned}(g_1) &= (c_2 + c_3) = (\beta) + (\gamma),\\ (g_2) &= (c_3 + c_1) = (\gamma) + (\alpha),\\ (g_3) &= (c_1 + c_2) = (\alpha) + (\beta).\end{aligned}$$

Since any three groups either contain four characteristics (α),

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* In the sense of Weber, Abel'sche Functionen, p. 19.

(β), (γ), (δ) in common, or any two have six in common (none of them paired), it follows that (α), (β), (γ) can be chosen in a variety of ways. The sum $(\alpha + \beta + \gamma)$ is either odd, (δ), or even. In the first case the theta functions ϑ_0, ϑ_1, ϑ_2, ϑ_3 of the first order and characteristics (0), (g_1), (g_2), (g_3) respectively vanish simultaneously for the half-periods α, β, γ. There are five linearly independent theta functions with $m = 2$ and characteristic (0) which vanish of order 2 for α, β, γ. We may take for these $x_i = \vartheta_i^2 (i = 0, \cdots, 3)$, $x_4 = \Theta_4$, the last being any one linearly independent of the other four. The points x_i lie on a spread S of degree 12, which has 61 quadruple points corresponding to the half-periods other than α, β, γ.

By equating to zero a theta function of the form $\Theta = \sum a_i x_i$ we obtain a surface of degree 12 in ordinary space. There are various types of such, according to the number and arrangement of the nodes of S which lie in the given flat. There are only four cases in which such a surface degenerates, namely, when $\Theta = x_i (i = 0, \cdots, 3)$. Then it becomes a sextic T counted double. The sextic, counted singly, may be represented by $\vartheta_i = 0$. There are 3 singular tangent planes and 25 double points, 8 in each of the singular planes, and one at their common point. By equating to zero other theta functions of the first order we obtain 24 twisted quartics of genus zero, 24 quintics, and 12 sextics lying on T.

II. If $(\alpha + \beta + \gamma)$ is even the groups

$$\begin{aligned}
(g_1) &= (\beta) + (\gamma) = (\sigma_1') + (\sigma_1'') = \cdots = (\sigma_5') + (\sigma_5''),\\
(g_2) &= (\gamma) + (\alpha) = (\sigma_1'') + (\sigma_1) = \cdots = (\sigma_5'') + (\sigma_5),\\
(g_3) &= (\alpha) + (\beta) = (\sigma_1) + (\sigma_1') = \cdots = (\sigma_5) + (\sigma_5'),
\end{aligned}$$

in which (σ_i), (σ_i'), (σ_i'') are odd, are necessarily of the form indicated, any two having six characteristics common. Otherwise they would all have four in common, whence it would follow $(\alpha) + (\beta) = (\gamma) + (\delta)$, (δ) being odd, which is contrary to the hypothesis.

The six theta functions with characteristics (0), $(\alpha + \sigma_i) = (\beta + \sigma_i') = (\gamma + \sigma_i'') (i = 1, \cdots, 5)$ vanish simultaneously for the half-periods α, β, γ.* These will be denoted by ϑ_0, ϑ_1,

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* It is evident that the three characteristics (α), (β), (γ) occur simultaneously in the five groups $(g_i) = (\alpha + \sigma_i)$ and only in these. This is a more precise form of Weber's theorem — "Irgend drei ungerade Charakteristiken, deren Summe gerade ist, kommen immer in mehreren Gruppen vor," *loc. cit.*, p. 22.

$\cdots, \vartheta_5$ respectively. Then $x_i = \vartheta_i^2 (i = 0, 1, \cdots, 4)$ determine a spread S' of degree 12 having 61 quadruple points. As in the preceding example, to each of the half-periods α, β, γ corresponds a two-spread (a two-dimensional locus) of degree 4.

Since two functions ϑ_i vanish simultaneously for 12 half-periods, three of which are α, β, γ, it follows that the flats $ax_i + bx_j = 0$ intersect S' in 9 of its nodes. If we require it to pass through a tenth not lying in either x_i or x_j, there are 29 points to choose from, and hence we determine 29.15 = 435 surfaces of degree 12, each having 10 quadruple points.

Every flat of the form $ax_i + bx_j + cx_k = 0$ contains 3 nodes of S' and can be made to pass through two other nodes by properly choosing the constants. There are thus determined 20.45 = 900 surfaces of degree 12 which have 5 quadruple points.

The coördinate flats $x_i = 0$ contain doubly covered sextics. Consider, for example, the sextic $\vartheta_0 = 0$, which we denote by T'. It has 25 nodes, 5 cubic curves $\vartheta_i = 0 (i = 1, \cdots, 5)$ lying in singular tangent planes, 18 quartic (genus zero), 30 quintic, and 10 sextic twisted curves.

CORNELL UNIVERSITY,
August, 1906.

ASSOCIATED CONFIGURATIONS OF THE CAYLEY-VERONESE CLASS.

BY DR W. B. CARVER.

(Read before the American Mathematical Society, September 3, 1906.)

In this paper, S_n will be used to denote a flat space of n dimensions; and the notation $C^{\nu}_{n,r}$ and the words "chiastic," "copoint," etc., will be used in the sense defined in the author's earlier paper * on these configurations.

Given five planes in S_3, we can, in general, construct the polar point P of any one of the five with respect to the 4-point determined by the other four planes. The five points so constructed, when joined by lines and planes, give a complete 5-point chiastic with the complete 5-plane; and the 5-point and 5-plane are said to be *associated*.*

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* "On the Cayley-Veronese class of configurations." *Transactions Amer. Math. Society*, vol. 6, pp. 534–545 (October, 1905).

Projecting this entire figure upon a plane (by what Veronese † calls an *univoca*, and his German translators an *eindeutige* projection) we have the following:

A $C^3_{5,2}$ (the Desargues configuration of two perspective triangles with their center and axis of perspective) breaks up in five ways into a $C^3_{4,2}$ (a complete 4-point) and a $C^2_{4,2}$ (a complete 4-line). Each $C^2_{4,2}$ and the corresponding $C^3_{4,2}$ determine a point P, which may be called the polar point of the $C^2_{4,2}$ with respect to the $C^3_{4,3}$; and the five points so determined, when joined by lines, form a $C^4_{5,2}$ (a complete 5-point) chiastic with the $C^3_{5,2}$. This $C^3_{5,2}$ may be said to be associated with the $C^4_{5,2}$.

This simple conception may be doubly extended, and we obtain the following general theorem:

Let there be given, in S_r, a $C^n_{n+2,r}$. This $C^n_{n+2,r}$ breaks up in $n+2$ ways into a $C^n_{n+1,r}$ and a $C^{n-1}_{n+1,r}$, and we can construct the polar point of each $C^{n-1}_{n+1,r}$ with respect to the corresponding $C^n_{n+1,r}$. The $n+2$ points so constructed give a $C^{n+1}_{n+2,r}$ chiastic with the $C^n_{n+2,r}$, and we may speak of the two as being associated.

It remains now to give a method for constructing the polar point of the $C^{n-1}_{n+1,r}$ with respect to the $C^n_{n+1,r}$. This can be done, without going outside of the space S_r in which the configurations lie, in the following manner:

Let the $C^n_{n+1,r}$ be lettered with $n+1$ letters, as described in the author's earlier paper. Any $n-1$ letters denote a line of the $C^n_{n+1,r}$ and on this line lies a point of the $C^{n-1}_{n+1,r}$. The point of the $C^{n-1}_{n+1,r}$ has a polar point P_{n-1} with respect to the two points of the $C^n_{n+1,r}$ lying on that line. (Let this P_{n-1} have a subscript made of the $n-1$ letters belonging to the line.) Any $n-2$ letters denote a triangle, or $C^2_{3,r}$, of the $C^n_{n+1,r}$. If we join the P_{n-1} on each line of such a $C^2_{3,r}$ to that point whose symbol contains the $n-2$ letters of the $C^2_{3,r}$ but not the remaining letter of the P_{n-1}, the three such joins meet in a point P_{n-2}. In the $C^n_{n+1,r}$ $n-3$ letters denote a $C^3_{4,r}$. If we join the P_{n-2} of each $C^2_{3,r}$ in such a $C^3_{4,r}$ to the proper point (determined as before), these four joins meet in a point P_{n-3}. * * * Finally, if we join each P_1 to that point of the $C^{n}_{n+1,r}$ whose symbol does not contain the letter of the P_1, these $n+1$ joins meet in a point P which is the polar point of the $C^{n-1}_{n-1,r}$ with respect to the $C^{n}_{n+1,r}$.

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* Cf. Morley, "Projective coordinates," *Transactions Amer. Math. Society*, vol. 4 See p. 292.

† Cf Grundzüge der Geometrie, p. 614.

The dual theorem for S_r gives a polar copoint, or S_{r-1}, of a $C^{r+1}_{n+1,r}$ with respect to a $C^{r}_{n+1,1}$; and, when $r=2$, we have, in the plane, a polar line of a $C^{3}_{n+1,2}$ with respect to a $C^{2}_{n+1,2}$. This special case in the plane is treated by Caporali, but his entire method is different from the method used in this paper.

If, having constructed in S_3 the polar point P of a plane with respect to the 4-point determined by four other planes, we project the whole figure upon a plane, *using P as the center of projection*, we obtain a $C^{3}_{4,2}$ and a $C^{2}_{4,2}$ which are associated, and the polar point of the former with respect to the latter is evidently indeterminate. In general, suppose we take, in S_n, the polar point P of a copoint, or S_{n-1}, with respect to the $(n+1)$-point determined by $n+1$ other copoints, and then project this figure upon an $S_r (r < n)$ from a co-S_r, or S_{n-r+1}, taking an S_{n-r+1} incident with the point P. We will thus obtain in S_r a $C^{n}_{n+1,r}$ and a $C^{n-1}_{n+1,r}$ which are associated; and the polar point of the $C^{n-1}_{n+1,r}$ with respect to the $C^{n}_{n+1,r}$ is indeterminate. *Whenever* a $C^{n}_{n+1,r}$ and a $C^{n-1}_{n+1,r}$ are associated, the polar point of the latter with respect to the former is indeterminate. This fact also Caporali notes for the plane case.

This polar idea for configurations admits of one more extension. If we have, in S_r, a $C^{n-k}_{n+1,}$ chiastic with a $C^{n}_{n+1,r}$ $(k \leqq r-1)$,[*] we may define the polar S_{k-1} of the $C^{n-k}_{n+1,}$ with respect to the $C^{n}_{n+1,r}$ thus:

There are ∞^{k-1} $C^{n-1}_{n+1,r}$'s chiastic[†] with the $C^{n}_{n+1,r}$ and also with the $C^{n-k}_{n+1,r}$, and any k of these $C^{n-1}_{n+1,r}$'s are sufficient[‡] to determine the $C^{n-k}_{n+1,}$. Construct the polar point of each of k of these $C^{n-1}_{n+1,r}$'s with respect to the $C^{n}_{n+1,r}$ and these k polar points will lie upon an S_{k-1} which is the polar S_{k-1} of the $C^{n-k}_{n+1,r}$ with respect to the $C^{n}_{n+1,r}$.

There is of course a dual theorem which defines a polar S_{r-k} of a $C^{r+k}_{n+1,r}$ with respect to a $C^{r}_{n+1,r}$.

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[*] Author's paper, *loc. cit.*, paragraph under theorem II.

[†] *Loc. cit.*, theorem VI.

[‡] *Loc. cit.*, theorem V.

VON HELMHOLTZ.

Hermann von Helmholtz. Von L. KOENIGSBERGER. 3 Bde., 8vo. Braunschweig, Vieweg und Sohn. Bd. 1: 1902, xi + 375 pp., 3 portraits; Bd. 2: 1903, xiv + 383 pp., 2 portraits; Bd. 3: 1903, ix + 142 pp., 4 portraits.

Vorlesungen über theoretische Physik. Von H. VON HELMHOLTZ. 6 Bde., large 8vo. Leipzig, J. A. Barth. Bd. 1_1: *Einleitung zu den Vorlesungen über theoretische Physik*, hrg. von KÖNIG und RUNGE, 1903, 50 pp., 1 portrait; Bd. 1_2: *Vorlesungen über die Dynamik discreter Massenpunkte*, hrg. von KRIGAR-MENZEL, 1898, x + 380 pp.; Bd. 2: *Dynamik continuirlich verbreiteter Massen*, hrg. von. KRIGAR-MENZEL, 1902, viii + 247 pp.; Bd. 3: *Vorlesungen über mathematische Principien der Akustik*, hrg. von KONIG und RUNGE, 1898, x + 256 pp.; Bd. 4: *Elektrodynamik*, "soll bald erscheinen"; Bd. 5: *Vorlesungen über die elektromagnetische Theorie des Lichtes*, hrg. von KONIG und RUNGE, 1897, xii + 370 pp.; Bd. 6: *Vorlesungen über Theorie der Wärme*, hrg. von F. RICHARZ, 1903, xii + 419 pp.

HELMHOLTZ died on September 8, 1894. Although he was at that time considerably past seventy years of age, he was still in the midst of his researches, still as enthusiastic as ever, still as piercing of intellect. On July 9, just before his last illness, he had been busy correcting the proof of his pupil Hertz's Mechanics, but nevertheless had stated that it had been a lucky day for him because he had discovered what he and many others before him had long been looking for. What this was we shall never know; for before he had had the necessary time to write the matter out and present it to the Academy he was mortally ill. His own researches and those of his pupil Hertz, who had died so shortly before, were by no means the only things that occupied his mind. He was director of the Reichsanstalt and had come to the decision two years previous to publish his lectures on mathematical physics. Indeed the fifth volume, on the electromagnetic theory of light, was already in the press with the proofs two-thirds read. This was a fitting end to the activity of one who for fifty years has been astonishing the world with the fecundity of his investigations in widely different fields of science. As Lord Kelvin said in an address before the Royal society in November 1904: "Of the whole of Helmholtz's great and splendid work in physiology, physics, and mathematics, I doubt whether any one man may be qualified to speak with the power which knowledge and understanding can give."

Upon some of Helmholtz's pupils devolved the task of seeing his lectures through the press; and at the deathbed of the great physicist's youngest son, when the name of von Helmholtz was slipping away from the world forever, his old friend and long time colleague at Heidelberg, Koenigsberger, determined to write his biography. The biography and the lectures (with the exception of the fourth volume) are now complete; and with the books which Helmholtz himself published they form a readily accessible monument which some of us may believe a more impressive memorial to his genius than that unveiled amid such royal pomp in the garden of the University of Berlin.

Koenigsberger's biography is not merely an account of Helmholtz's private life, it is not an analysis of his scientific achievements, it is not the memoirs of an intimate acquaintance nor the critical history of an outsider; it is all of these together. On one page the reader will find some family correspondence, perhaps concerning the health of Robert or Fritz; on the next, will be a critique of a series of articles on the foundations of geometry or sometimes an unprinted public address. As one reads, he follows the whole life of the master.

The work is divided into sections corresponding to the different positions which Helmholtz occupied — a student, a naval surgeon, teacher in an academy of fine arts, professor of physiology at Königsberg, professor of anatomy at Bonn, of physiology at Heidelberg, of physics at Berlin, and president of the Reichsanstalt. The titles alone should seem to justify Kelvin's remark. It should be remembered that Helmholtz did not have a free choice as to his profession. He always said that he was a born physicist. But his father was poor; being a physicist was too dear; the boy must choose medicine as his work even though both he and his father regretted it. The result was one of the world's greatest physiologists, with the Lehre von den Tonempfindungen and the Physiologische Optik as chief witnesses. He could not have been a greater physicist even if he had not been a physiologist, perhaps not so great.

It would be ungracious to draw any moral concerning the elective system of studies and the necessity of doing what one pleases too early in life.

Helmholtz had a very marked ability for keeping out of ruts. A predisposition for getting snugly into some rut is perhaps one reason for the opinion current in some quarters to the effect that one's work is done at the advanced age of forty. If Helmholtz had lived no longer, we should have had the Erhaltung der Kraft, the Integrale der hydrodynamischen Gleichungen welche den Wirbelbewegungen entsprechen, and the Handbuch der physiologischen Optik; but what of the Lehre von den Tonempfindungen and indeed the last third of the Physiologische Optik, what of the work on the foundations of geometry which led to Lie's researches on that subject, and the investigations on electromagnetism which culminated in Hertz's discoveries, or the Thermodynamik chemischer Vorgänge which, written at the age of sixty, would have been the great landmark in physical chemistry if Gibbs's then unknown work had not appeared half a dozen years before? And this is not all. It is necessary to read the whole biography to appreciate how Helmholtz retained his spontaneity and power into the last days of his life. A rare parallel to this is seen in the life of his close friend, Kelvin.

In its general make-up as a book the biography is almost de luxe. The paper is heavy and the type, which fortunately is Roman, is clear. The reproductions of portraits, for the most part by Lenbach, add to the artistic appearance of the work, and to its personal interest. The scientific world has reason to thank the publisher for the form in which the volumes have appeared just as it should be very grateful to the author who, though by no means a young man, has devoted so much time to accomplish religiously his self-appointed task.

Helmholtz's Lectures on theoretical physics are particularly interesting and instructive because he himself was personally responsible through his own researches for so large a part of their contents. It is this which gives them their life and individuality. Moreover the acquaintance of the author with so great a realm of physics, mathematics, and biology gives a virility to his lectures and a powerful variety to his illustrations which it would be difficult to find elsewhere.

The introduction of fifty pages which forms the content of the first part of the first volume is a sort of epistemology or methodology of the physical sciences. This subject was of great interest to Helmholtz and upon it he had printed a large number of papers which may be found admirably analyzed in the biography. The author was too much of a physiologist to be won over without restriction to that abstract formalism which was represented by Thomson and Tait and by Kirchhoff. He doubtless could understand their point of view when they undertook to demonstrate that force did not exist; but he preferred to retain force as a fundamental idea. In the treatment of the number system the same practical point of view appears. Note the statement: "That diagonals of squares exist in nature there is no question; and so, too, definite physical investigation may thrust other irrationalities upon us. The difficulty or the impossibility arises here, not in regard to the reality of such quantities or ratios, but in the incompleteness of our methods. . . . Similarly with the much discussed question of the existence of continuous quantities." This and other statements which the author makes may furnish a sounder basis for metaphysical speculation than many a more complete theory developed by those less in touch than he was with the real living world.

Such careful attention to the underlying conceptions is everywhere typical of the lectures. The discussion of the dynamics of discrete masses begins with a detailed account of such things as the material point, coördinates, continuity and differentiability, velocity, acceleration, Newton's laws, and so forth. It would be difficult to find a clearer and more careful statement of these fundamental matters. As might be expected from the director of the Reichsanstalt, the theory of pendular vibrations is treated in great detail, sixty-five pages being given to it. To illustrate the way in which the author gives the student the necessary mathematical methods as he goes along we may mention that the questions of least squares and of linear non-homogeneous differential equations of the second order find their treatment at the points where each is needed. Stress is laid on the importance of the general integrals of motion, the integrals of momentum, of moment of momentum, and of energy. Work and energy receive the attention which one might expect to be given them by the author of the Erhaltung der Kraft. The various fundamental principles such as that of virtual velocities, of d'Alembert, and of Hamilton, and the lagrangian equations of motion are all taken up. It may be noted that Appell in his Mécanique rationnelle follows this same plan of introducing into dynamics at a somewhat early stage these higher principles. It would be difficult to find an elementary treatment of mechanics which could better serve as an introduction to mathematical physics than this of Helmholtz.

Before taking up the theory of continua, the author gives a short discussion of the relation of the theory to the molecular hypothesis, and of the difference between ordered and unordered motion in the interior of the continuum. He then establishes a connection with the homogeneous strain by developing the displacement into a Taylor's series and examining the effect of the terms of higher order. After a preliminary treatment of strain, he makes the statement that the internal forces in a continuum follow the law of action and reaction and hence are conservative. This leads the way to Green's method of developing the theory of elastic bodies from the existence of a potential for the forces of deformation. The analytic theory of elasticity is beset with massive formulas. These the author largely avoids by turning his attention to the more physical side of the question. He sets about determining the function whose space integral gives the potential energy in a variety of different types of elastic bodies. The following section of the book treats of the equilibrium of elastic bodies. Those cases are solved which are of use in actually determining the elastic constants of the bodies. Finally the question of the propagation of plane or spherical waves, whether longitudinal or transversal, is taken up and solved. Helmholtz was about seventy-three years old when he prepared and delivered these lectures and must have been, if the lectures may be taken in evidence, an admirable teacher; for he shows a genius in avoiding analytic difficulties wherever he can, and in resolving them clearly for the student when to avoid them is impossible or undesirable.

With the third volume, which is on sound, there begins to appear a considerable amount of repetition. The amount of attention paid to the motion of a pendulum was noted in speaking of the first volume; here the matter must needs come up again. So different, however, is the method of procedure and so much is the original problem generalized that the repetition seems advantageous rather than otherwise. It is a serious mistake to avoid altogether the repetition of matters and methods which are so fundamental that they should become second nature to the student. It is not possible to be old friends at first sight. Of course Fourier's series are the foundation of a large part of the mathematical work. Instead of assuming that they are known or may be learned by reference to some standard text, the author patiently and in all detail develops the method of Fourier's series, enters upon the necessary proofs of convergence and even goes so far as to show that the convergence is toward the right value. This development of the mathematical algorithms as they are needed goes a long way toward giving the reader that confidence which is so important. In treating waves in space, Green's theorem and Huyghens's principle are taken up in the same manner. In discussing vibrations, each different type, whether a plucked string, a stretched string, a membrane, free air, or air confined to a tube, is developed separately and in natural order.

The fourth volume, on electrodynamics, has been announced as "soll bald erscheinen" for almost a record breaking number of years. It is even rumored that the volume may never appear. This is probably due to the great changes which have taken place in electromagnetic theory since the lectures were delivered. The editors may feel that there would be little use in printing a book which would be out of date before it was written. It seems to us that this reason should not prevail here. Helmholtz's researches on electricity and magnetism were by no means insignificant. A large part of the time of his later years was spent in this field of physics. It was he who developed the theory of an ether which included Maxwell's as a special case. His lectures on electrodynamics could not fail to be of interest as a matter of history if in no other way. Furthermore it is inconceivable that what Helmholtz might have to say on the subject, his methods of attack, and his points of view should not be highly important and instructive even after the world has moved somewhat away from his position. We hope the volume may still appear.

The fifth volume, that on the electromagnetic theory of light, was the first to be published. Fortunately it is independent of the preceding one. With characteristic straightforwardness an introduction on some points in electrical theory and on waves in a continuous medium comes before the introduction of the Maxwell equations. In their proper places the elements of the theory of the potential function, Green's theorem, plane and spherical waves, and Huyghens's principle reappear, although each has been taken up at least once before. After a long presentation of diffraction and the establishment of the laws of geometric optics from the fundamental dynamical equations, there follows a detailed treatment of the customary questions of geometric optics. Polarization and dispersion are not neglected; but the fact that the work is one on general physical optics rather than on special mathematical problems in optics is clearly indicated by the meagre attention paid to crystalline media — only eight pages. This is but another instance of the author's avoidance of complicated mathematics and insistence upon that which will be of most value to the practical physicist.

The lectures on the theory of heat are a charming close to the whole series. The introduction on "warmness" and temperature reminds one irresistibly of the author's philosophical speculations with which the series began. The treatment of problems in conduction with the repetition of the necessary mathematical tools, Green's theorem and Fourier's series, not omitting even the proofs of convergence, recalls the general style of the other volumes. The applications of these results are, however, new and lead up to the introduction of Fourier's integral with a full explanation of what this new tool does for one. Throughout the volume, the illustrative examples have even more than their accustomed variety — cooling of the earth, diurnal and annual variations in subterranean temperature, the vital theory of the black body, and Kirchhoff's law. The next large division of the volume concerns thermodynamics. As might be expected, the underlying concepts, the first law or conservation of energy, the second law or dissipation of energy, the entropy, the "free" energy, and so on, are set forth with especial pains. It would be difficult for the reader to fail to get the right idea. It is here and in the following section on cyclic systems that we find some account of the researches which took the major part of the author's free time after passing the age of sixty. The work appears to be still new to him. There is a freshness about the style which is not so much in evidence in the volumes which deal with his earlier work.

It has not been our intention here to give an account of Helmholtz's life and works on mathematical physics — this were too great a task — but to indicate as briefly as might be what may be found in the eight or nine volumes which treat of these subjects and which appeared during the decade after his death. Of the lectures Koenigsberger says: "(Sie) bilden jetzt die ausgezeichnetsten Lehrbücher einzelner Theile der mathematischen Physik." Nothing could be truer. And may we not in all seriousness ask the question whether these live works of a master genius may not be more inspiring and really better text-books for the student than the work of some lesser person, no matter how careful and industrious he be? Of the biography let us add that it ends with consummate inspiration in the words which Helmholtz himself applied to Goethe and Beethoven : "Wir verehren in ihnen einen Genius, einen Funken göttlicher Schöpferkraft, welcher über die Grenzen unseres verständig und selbstbewusst rechnenden Denkens hinausgeht. Und doch ist der Künstler wieder ein Mensch, wie wir, in welchem dieselben Geisteskräfte wirken wie in uns selbst, nur in ihrer eigenthümlichen Richtung reiner, geklärter, in ungestörterem Gleichgewichte, und indem wir selbst mehr oder weniger schnell und vollkommen die Sprache des Künstlers verstehen, fühlen wir, dass wir selbst Teil haben an diesen Kräften, die so Wunderbares hervorbrachten."

EDWIN BIDWELL WILSON.

YALE UNIVERSITY, NEW HAVEN, CONN.,
October 8, 1906.

PIERPONT'S THEORY OF FUNCTIONS.

The Theory of Functions of Real Variables, Volume I. By JAMES PIERPONT, Professor of Mathematics in Yale University. Boston, Ginn and Company, 1905. xii + 560 pp.

SINCE the time of Weierstrass the so-called rigorous style in mathematical writing has increased constantly in favor until in recent years it has become a commonplace instead of a rarity. Such myriads of microscopic ϵ's and δ's have penetrated our mathematical thinking that it would be impossible to rid the system of them entirely even if it were desirable to do so. A return to the externals of the intuitive style of writing would, however, enable any one individual to obtain a conception of a much wider range of investigations, and it seems possible that such a return may be made after mathematicians have come to an understanding acknowledged by all of the precise meaning of the various notions of continuity and limits used in analysis. Many of the details now necessary in a rigorous exposition could then be omitted. Before this can happen, however, books of the general character of the one with which this review is concerned must become familiar reading.

Professor Pierpont's book is the first of two volumes in which he will present the subject matter of his lectures at Yale University on advanced calculus and the theory of functions of real variables. The contents may be roughly divided into three parts, the first on the real number system and the real exponential and logarithmic functions, the second on differential calculus with chapters on point aggregates and implicit functions, and the third on integration, including a more than usually complete discussion of improper integrals and integrals involving a parameter. The purpose which the author had in mind is made clear in one of the paragraphs of the preface. After a statement that the American student about to enter into graduate work often understands much of the formal processes of the calculus without having a clear knowledge of the conditions under which their applications are justifiable, he writes:

"The problem therefore arises to examine carefully the conditions under which the theorems and processes of the calculus are correct and to extend as far as possible or useful the limits of their applicability."

Some of the chapters deserve especial mention on account of their clearness and finish. For example, it would be hard to find more attractive accounts of the theory of implicit functions, indeterminate forms, and the definition and properties of a definite integral than are given in Chapters IX, X, and XII, respectively. On the other hand other parts of the book impress one as being relatively unfinished and somewhat overcomplicated. Among these are the sections devoted to integrals involving a parameter, which will be mentioned later. The pedagogical principle that clearness is attained by proving special cases of theorems first, instead of treating them as corollaries to the general case, has caused the book to be larger than it would otherwise have been, and it is doubtful if after all the advantage gained is very great.

Among the most interesting features of the book are the sections entitled "Criticism." As the author states in the preface, nothing stimulates the student's critical sense more powerfully than asking him to detect the flaws in a piece of reasoning which at an earlier stage of his training he considered correct. It is especially interesting, therefore, to find after each important discussion, sections in which the inaccuracies of the more elementary treatises are exposed. Mention may also be made here of the illustrative examples and the special theorems on limits. The former are numerous and well chosen, often being constructed to show how a theorem may fail to be true when its initial hypotheses are not satisfied. The latter occur especially in Chapters II, III, and VI, where many useful inequalities and limits are calculated.

In Chapters I and II Professor Pierpont has developed the theory of the real number system in a way which can not fail to be of interest. Supposing the theory of the system I of positive integers to be known, he introduces in succession three new systems, F, R, $\mathfrak{R}$, the first consisting of elements (a, a') with the properties of positive rational fractions, the second of elements $\{a, a'\}$ with the properties of the differences $a - a'$ of positive rational numbers, the third being the system of all real numbers. It is seen that the system I is closed only under addition and multiplication, while F is closed under addition, multiplication and division. R and $\mathfrak{R}$ are closed under all the rational operations and $\mathfrak{R}$ has the additional property of admitting limiting processes. Each new system consists of entirely new elements but in each there is a subsystem with elements and properties in correspondence with those of the one just preceding.

In Chapter II, the irrational number is defined after Cantor to be * a sequence of rational numbers

$$(a_1, a_2, a_3, \cdots, a_n, \cdots)$$

having the property that for any $\epsilon > 0$ there exists a positive integer m such that $| a_n - a_\nu | < \epsilon$ whenever n and ν are greater than m, or in Professor Pierpont's concise and satisfactory notation, a sequence for which

$$\epsilon > 0, m, \quad | a_n - a_\nu | < \epsilon, \quad n, \nu > m.$$

The totality of such sequences is the real number system $\mathfrak{R}$.

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* Professor Pierpont writes on p. 36, "every regular sequence *defines a number.*" The difference between saying that the sequence *is* a number and saying that it *defines* a number, is slight. The former is perhaps less mysterious.

The sections (§§ 112–123) in which it is pointed out that the numbers of $\mathfrak{R}$ and the points of a line are in one-to-one correspondence are not very convincing because to make a logical demonstration of this fact it would be necessary to state explicitly the postulates which define a geometric continuum. As far as the applications in other parts of the book are concerned it would have sufficed to show that the points of a line can be used as a kind of notation for the real number system. Indeed much of graphic mathematics which seems intuitive would be precise if the graphs were regarded merely as notations for ideas elsewhere sharply defined. In the latter part of the chapter the definition of a Dedekind partition of the rational numbers is defined and the correspondence of these to the numbers of $\mathfrak{R}$ is shown.

Taken altogether, Professor Pierpont's treatment of the number system is a very suggestive one, especially valuable from the pedagogic standpoint. There are a few places where improvement seems possible. On page 24 one might infer that the properties 1°, 2° characterize a sequence. That this is not true is shown by the example*

$$a_1, a_2, a_3, \cdots, a_n, \cdots; b_1, b_2, \cdots, b_n, \cdots,$$

which has the properties referred to. These properties, however, have but slight importance in the following pages, as the notion of sequence actually used has always been a set of elements in one-to-one correspondence with the positive integers. In connection with the definitions of order and the algebraic operations on irrational numbers some essential points have been omitted. It should be shown (page 39) that any two numbers α, β satisfy one and but one of the relations $\alpha \lesseqgtr \beta$. Inasmuch as the same number has an infinity of different representations, the discussion is hardly complete without a proof that the order relations and the result of any algebraic operation are independent of the particular representations chosen for the numbers involved. Lack of space would render it inadvisable to give a complete treatment of the number system from the standpoint of the theory of assemblages, but in the sections in which the properties of R and $\mathfrak{R}$ are discussed it would have been easy to point out especially those which as postulates characterize these systems categorically.

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* Huntington, "Types of serial order," *Annals of Math.*, 2d series, vol. 6, p. 169.

In Chapter III the theory of the exponential a^z is based upon the equation $x^n = a$, n a positive integer. This equation determines uniquely a Dedekind partition which defines a unique position root ρ, designated $a^{1/n}$. The definition of a^z when z is any rational number follows immediately, and when z is an irrational number $(r_1, r_2, \cdots, r_n, \cdots)$, a^z is the number $(a^{r_1}, a^{r_2}, \cdots, a^{r_n}, \cdots)$. Similarly the theory of the logarithm is based upon the equation $b^x = a$.

With Chapter IV begins the part of the book devoted to the theory of functions. Like Chapter V it is largely descriptive in character, the one being devoted to the explanation of the fundamental notions and notations concerning functions, and the other serving the same purpose for point aggregates. Attention need only be called to an interesting but much abbreviated proof that the trigonometric functions are not algebraic, to a very useful distinction between never decreasing (increasing) or monotone functions and always increasing (decreasing) or univariant functions, and to the notations $D(a)$, $V(a)$ introduced on page 153. By $D_\rho(a)$ is meant the neighborhood $a - \rho \leqq x \leqq a + \rho$ of the point a, and when only the points of a particular aggregate $\mathfrak{A}$ are under consideration, $V(a)$ stands for the points of $\mathfrak{A}$ which are in $D(a)$. The symbols $D^*(a)$ and $V^*(a)$ have the same meanings except that in each case the point a itself is excluded. The definition of the limit of a function may now be stated: the function $f(x)$ defined over an aggregate $\mathfrak{A}$ is said to have a limit η as x approaches a, if for any $D(\eta)$ there exists a $V^*(a)$ with the property that whenever x is in $V^*(a)$ the value of $f(x)$ is in $D(\eta)$. It is evident that this definition holds also when η or a is infinite, provided that proper meanings are assigned to $D(+\infty)$ and $V(+\infty)$. One needs only to notice the places in Professor Pierpont's book where many special cases of theorems on limits have been or might have been collected in a single statement and proved with a single proof by means of these notations, in order to understand how really serviceable they may be. At the end of the chapter, Cantor's example of an aggregate which is perfect in an interval $\mathfrak{A}$ but does not contain all the points of $\mathfrak{A}$, is given.

So far no mention has been made in the book of limits of functions. The fundamental definition given first in Chapter VI is what may be called the sequence definition. A function $f(x)$ is said to have the limit η as x approaches a if the values

$f(a_n)$ $(n = 1, 2, \cdots)$ have the limit η for every sequence a_n such that $a_n \neq a$ and $\lim a_n = a$. This definition has the advantage that it can be applied when η or a is infinite, and the other well-known criteria for the existence of a limit given in the succeeding sections are readily derived from it. It is to be regretted that the author has not made more use of the notations $D(a)$, $V(a)$ described above. Much space would have been saved here (§ 280–283) and in later proofs, as for example that of § 324, by grouping all the special cases involving infinites into a general one. The sections explaining the geometric interpretations of the criteria for limits are very helpful. The discussion of iterated limits is brief, only one criterion being given, although the later chapters contain many instances of iterated limits of various kinds for which special criteria have been derived in each case. Perhaps one reason why these have not all been treated by similar methods is that the criterion given by Professor Pierpont demands the existence of the double limit

$$\underset{\substack{x=a \\ y=b}}{\mathrm{L}} f(x, y),$$

which for example in the theorem

$$\int_a^b \frac{\partial f(xy)}{\partial y}\, dx = \frac{\partial}{\partial y} \int_a^b f(xy)dx$$

is not of primary interest. In this connection attention may well be called to a theorem of Professor E. H. Moore* to the effect that the two iterated limits

$$\underset{y=b}{\mathrm{L}}\ \underset{x=a}{\mathrm{L}} f(x, y) \text{ and } \underset{x=a}{\mathrm{L}}\ \underset{y=b}{\mathrm{L}} f(x, y)$$

exist and are equal, provided that the two inner limits exist, and that one of them, say $\underset{x=a}{\mathrm{L}} f(x, y)$, is approached uniformly in the neighborhood of $y = b$.

On pages 202–205, after a section on uniform convergence, some curious and interesting analytical representations of special functions are given. For example, the function of x whose value is $+1$ for $x > 0$, -1 for $x < 0$, and 0 for $x = 0$, denoted by signum x, is represented in the form

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* Lectures on advanced integral calculus, The University of Chicago, autumn quarter, 1900; unpublished. The theorem was also stated at the Colloquium of the American Mathematical Society, held in New Haven, Conn., Sept. 5–8, 1906.

$$\operatorname{sgn} x = \frac{2}{\pi} \lim_{n=\infty} \operatorname{arc\,tan} nx.$$

The often-quoted function of Dirichlet, which is equal to a for rational values of x, and to b for irrational values, has then the representation

$$y = a + (b - a) \lim_{n=\infty} \operatorname{sgn} (\sin^2 n!\, \pi x).$$

After the notion of limit of a function has been made clear, the definition of a continuous function is an easy matter. In Chapter VII properties of a continuous function, such as the fact that it takes on all values between its upper and lower limits, are derived in a very attractive way, and examples are given illustrating various kinds of discontinuties.

The title of Chapter VIII is "Differentiation." The derivative is defined as usual in the more elementary books without the use of the right and left upper and lower derivatives of Dini, although the examples given of functions not derivable at certain points would serve well to illustrate them. The rules governing the use of derivatives and differentials are derived with especial care and the weaknesses common in the calculus text-books are criticised. In the conclusion of the theorem on Taylor's formula (page 255), the word "in" in the phrase "Then for any x in $\mathfrak{A}$", should be replaced by within, according to the author's usage. For the expression on the right hand side of equation (1) is not well-defined at $x = a$ on account of the derivative $f^{(n)}(c)$ which occurs in it. In the latter part of the chapter, devoted to functions of several variables, the theorem on the reversibility of differentiation for a function $f(x, y)$ could be much improved. Under the assumption that f'_x exists on $y = b$, f'_y on $x = a$, and that one of them is approached uniformly, it follows as a corollary to the theorem of Moore mentioned above, that the second derivatives f''_{xy}, f''_{yx} exist at (a, b) and are equal. The hypotheses which Professor Pierpont makes in order to prove the theorem refer to a region about the point (a, b), as must necessarily be the case if one uses the law of the mean as he does in the proof. A similar remark applies to the theorem of page 271 in which the conditions are set down for a function $f(x, y)$ to be totally differentiable, that is, to be expressible in the form

$$f(a + h, b + k) = hf'_x(a, b) + kf'_y(a, b) + \alpha h + \beta k,$$

in the neighborhood of the point (a, b), where α and β approach zero with h and k. The proof can in fact be made without using the law of the mean on the assumptions that on $x = a$ the derivative f'_x exists, is approached uniformly, and at $y = b$ is continuous, while the derivative f'_y exists at (a, b). After the proof of Taylor's formula for functions of several variables an interesting example is given, which illustrates a case in which the expansion does not hold.

The method applied in Chapter IX to implicit functions is a method of approximation due to Goursat. It is somewhat artificial but has the advantage of demanding the existence of the derivative with respect to the dependent variable only. Usually for an equation in the form

$$F(x, u) = 0,$$

the hypothesis is made that the function F has both of its first partial derivatives continuous. Professor Pierpont's is apparently the only one of the standard books in which only the derivative with respect to u is required. Systems of equations are, however, treated under the usual assumptions.

The following chapter on indeterminate forms is an exceptionally valuable one because it has to do with a subject which is badly treated in the elementary books and neglected in the more advanced treatises. The theorems are clean cut and stated with a minimum of hypothesis. In the theorem of § 452 the neighborhood $RD(a)$ should evidently be $RD^*(a)$, and at the bottom of the page the proof might have been simplified by merely referring to the theorems on the limits of a product and sum. In one or two places several theorems could have been combined into one (*e. g.* §§ 450, 451, and §§ 452, 453) with the help of the author's notation for neighborhoods, already mentioned a number of times.

Chapter XI on maxima and minima is noteworthy for the generality of the theorems for functions of a single variable, and for the pointed way in which one is warned against errors common in discussing maxima and minima of several variables. The following theorem will serve to illustrate the first point:

"*Let $f(x)$ be continuous in $D(a)$. In $D^*(a)$ let $f'(x)$ be finite or infinite, and never vanish throughout any interval of it. In $RD^*(a)$, let $f'(x)$ be positive when not zero; in $LD^*(a)$, let $f'(x)$ be negative when not zero. Then $f(x)$ has a minimum at a.*"

On pages 317 and 321, Professor Pierpont urges the reader to distinguish carefully between the terms maximum of a function at a point, and maximum of a function in an interval. It would be better to use an entirely different word, for example upper limit or upper bound, to designate the latter. The use of the word *extreme* for either a maximum or a minimum is a helpful simplification in many places. The theorem of § 476 on extremes of functions of several variables might have been proved as briefly by a method similar to that used for functions of one variable, in which only the existence and continuity of the derivatives of the nth order, *i. e.*, the first not vanishing identically, is assumed. The examples and criticisms of §§ 479–483 bring out clearly the necessity for care in the case of functions of several variables.

The last five chapters of the book are devoted to the theory of simple proper and improper integrals, and multiple proper integrals. The first of the five is introductory and contains the usual Riemann definition of a definite integral with criteria for integrability and a discussion of certain classes of functions which are integrable. A later part of the chapter is devoted to the theory of content of a linear point aggregate $\mathfrak{A}$ from a standpoint similar to that of Jordan, and to a generalization of the definition of a definite integral by means of which it is possible to integrate a function $f(x)$ over a point set $\mathfrak{B}$ as well as over an interval. The definitions of the "outer content" of Pierpont and of "l'aire extérieure" of Jordan are different, the former being the lower bound of all intervals containing points of $\mathfrak{A}$, while the latter is the lower bound of all intervals containing points of $\mathfrak{A}$ or its boundary. The generalized integral is in fact the ordinary integral, over an interval (a, b) containing $\mathfrak{B}$, of an auxiliary function $g(x)$ which has the same value as $f(x)$ at points of $\mathfrak{B}$ but is elsewhere zero. The last two theorems of the chapter are stated as corollaries, but it is difficult to see that they follow easily from the ones immediately preceding.

In Chapter XIII further properties of proper integrals are developed. Among them one finds the two theorems of the mean, a proof of the fundamental theorem of integral calculus, and theorems on the change of variable in proper integrals.

The proof of the fundamental theorem on page 370 is one of the gems of the book, and it is to be hoped that the criticism immediately following will be widely read. The theorem of § 543 which states that the upper, and similarly the lower integrals of $f(x)$ and $f(\psi(u))\psi'(u)$, taken over corresponding intervals, are equal when $x=\psi(u)$ has a non-vanishing continuous derivative, is the key to the transformation theorems. From it the preceding theorem of § 540 follows as a corollary, and it is evident also that the part of the hypothesis in this theorem which assumes the integrability of $f(\psi(u))\psi'(u)$ is unnecessary. In fact it can be shown that if the functions $f(x)$ and $f(\psi(u))\psi'(u)$ are both integrable, one needs only to assume $\psi'(u) \geqq 0$ (or $\leqq 0$) instead of the stronger assumption $\psi'(u) \neq 0$. The theorem of § 540 applying to the case when $\psi'(u)$ is continuous but not otherwise restricted is notable for its generality and the elegance of the proof. The properties of integrals depending upon a parameter, including continuity, derivability, and integrability with respect to the parameter, are discussed in the concluding pages of the chapter. On page 387, Professor Pierpont has defined a function $f(x,y)$ to be regular in a rectangle $a \leqq x \leqq b$, $\alpha \leqq y \leqq \beta$, if for each particular value of y it is a limited integrable function of x in the interval $a \leqq x \leqq b$. It does not follow that $f(x, y)$ is limited in the whole rectangle, although for several of the proofs, notably those on pages 391, 392, such an assumption is necessary. On page 424, where the use of the word regular is still further restricted, the fact that $f(x, y)$ is limited follows from 1°. It is readily understood that the theorems mentioned above, concerning integrals involving a parameter, are really theorems on the inversion of iterated limits in which one or both of the limiting processes is an integral. Inversion is permissible in these cases under precisely the same conditions as for ordinary limits. For example in the equation

$$\lim_{y=\eta}\int_a^b f(x, y)dx = \int_a^b \lim_{y=\eta} f(x, y)dx,$$

the two outer limits exist and are equal if the two inner limits exist, and if one of them is approached uniformly. It is evident then that the hypothesis,

$$\lim_{y=\eta}\int f(x, y)dx$$

integrable, made in § 565 is superfluous. Similar alterations could be made in § 567 and § 569 where the derivative and integral with respect to y of

$$\int_a^b f(x, y)dx$$

are discussed. The inversion of integration is proved in the present chapter only for continuous functions, but in much more general form in the last chapter of the book.

The improper integrals considered in Chapters XIV and XV are those in which the integrand has a finite number of infinities, and those in which the interval of integration is infinite. The value of this part of the book lies chiefly in the thoroughness of the extension, to these two kinds of improper integrals, of the theorems well-known in the theory of proper integrals. Aside from the criteria for convergence, the properties developed are for the most part analogous to those given in the preceding chapter. At the end of Chapter XV applications are made to the B and Γ-functions. Naturally the hypotheses which must here be made in order to prove theorems on integrals involving a parameter are more complicated than for the simple integrals, and the question whether or not simplifications like those mentioned in connection with the last chapter are possible, could be answered only after careful investigation.

The final chapter of the book is on multiple proper integrals and contains properties analogous to those given in the earlier chapters for the simple integrals. As before there is a section on point aggregates, in which the properties of n-dimensional aggregates useful in connection with the multiple integrals are developed. The theory of the content of an aggregate $\mathfrak{A}$ is made to depend upon the fundamental properties of multiple integrals by means of an auxiliary function $f(x)$, which is equal to 1 at all points of $\mathfrak{A}$ and vanishes elsewhere. The upper and lower integrals of $f(x)$ in a field containing $\mathfrak{A}$ are the upper and lower contents of $\mathfrak{A}$ respectively, and when they coincide $\mathfrak{A}$ is said to be measurable. In § 722 it is shown that the notions of upper and lower multiple integrals may be generalized so that these terms have meaning when the field of integration is any limited point aggregate $\mathfrak{A}$. Furthermore the subdivisions of $\mathfrak{A}$ used in the definition need not be by planes parallel to the axes, but may be of the much more general character specified on page 521. The treatment of multiple integrals is somewhat more general than that of simple integrals in that many of the theorems relate to the upper and lower integrals without any assumption that the function involved is integrable. One of the principal problems of the chapter is the reduction of multiple integrals to iterated integrals. The two theorems of § 733 on iterated integrals are very general, and include as a corollary a similar theorem of Pringsheim* applying to an integrable function. One of the conclusions may be stated to give an idea of the nature of the theorems:

$$\underline{\int}_{\mathfrak{A}} f d\mathfrak{A} \leqq \int_{\mathfrak{x}_1} d\mathfrak{x}_1 \underline{\int}_{\mathfrak{P}_1} f d\mathfrak{P}_1 \leqq \overline{\int}_{\mathfrak{x}_1} d\mathfrak{x}_1 \int_{\mathfrak{P}_1} f d\mathfrak{P}_1 \leqq \overline{\int}_{\mathfrak{A}} f d\mathfrak{A}.$$

Here the dashes distinguish upper and lower integrals; $\mathfrak{x}_1$ is the projection of the aggregate $\mathfrak{A}$ on the x_1-axis and $\mathfrak{P}_1$ is the variable section of $\mathfrak{A}$ by the plane $x_1 = \text{const.}$, depending therefore upon x_1. The final conclusion with regard to the reversibility of integration of a function $f(x, y)$ in a rectangle, including all the cases considered in previous chapters, is that it is possible when f is a limited function whose discontinuities on any line parallel to one of the axes, and also as a whole, form a discrete aggregate. The chapter closes with theorems on the transformation of multiple integrals of any order. Hitherto these theorems seem to have been given in detail only for double integrals.†

G. A. BLISS.

PRINCETON,
October, 1906.

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* Encyklopädie der mathematischen Wissenschaften, II A 2, p. 105.

† See Pierpont, "On multiple integrals," *Transactions Amer. Math. Society*, vol. 6 (1905), p. 416

THE MATHEMATICAL TRIPOS OF 1906.*

Cambridge University Examination Papers, Easter term, 1906; Containing the Papers for the Mathematical Tripos, Parts I and II. Cambridge, at the University Press. Pp. 739–782. Price 2s. 6d. (For sale through Deighton, Bell and Company, Cambridge, England.)

THE main difference between the average British student of mathematics and his continental brother is summed up in the relative importance of examination and dissertation. While the examination for the German doctorate is a single oral ordeal lasting not over a couple of hours, the tripos for the first degree (baccalaureus artium) occupies several days, and the honor papers make demands on the successful candidates which could seldom be met by a young German doctor. On the other hand, the German student has been engaged on a single problem for one to two years, and his results make a more or less important contribution to the science. The student at Cambridge has concentrated all his skill on short, definite exercises, most of which, however, have a direct bearing on general mathematical development. The principal feature of the examinations is the solution of a large number of strictly original problems.

The names which have become famous among English mathematicians are almost identical with those which have stood high in the examinations, but many of these very men are now active in the movement to change their character by allowing time for the second part, and putting more emphasis on real original work, rather than on "an excessive amount of polishing of mathematical tools." †

The tripos papers of 1906 contain twenty examinations of three hours each. The number of questions varies from four to twenty, and these range in difficulty from a simple corollary in plane geometry to abstruse theorems on the outer boundary of mathematical knowledge. The papers are arranged in three groups. The first group of seven is set for all candidates for the B.A. degree with honors in mathematics. They are taken at the end of the third year of residence, but in exceptional cases may be taken at the end of the second year. The subject matter of each examination is announced in advance; it may include a number of divisions. The questions usually consist of two parts, the first being a general theorem, frequently found in the text books, while the second part is a problem, not generally known, and probably having only a distant connection with the first part. The topics of these examinations are: (1) Euclid and conics, elementary modern geometry; (2) algebra and trigonometry; (3) analytic geometry and elementary calculus; (4) statics and dynamics; (5) geometric optics and elementary astronomy; (6) hydrostatics, heat and electricity; (7) problem paper on subjects (1) to (6). It will thus be observed that pure and applied mathematics are given equal weight. In this first part quickness and dexterity are essential to success.

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* For an account of the development of the mathematical tripos, one should consult the History of the study of mathematics at Cambridge, by W. W. Rouse Ball, Cambridge, 1899, and the inaugural address before the London mathematical society by Dr. Glaisher, in 1888.

† See BULLETIN, volume 12, page 468.

After the first set of seven papers there is an interval of eleven days, at the end of which the list of honor men is published. These then take the second set of seven papers. On the result of the whole fourteen papers the candidates are arranged in order of merit as wranglers, senior optime, and junior optime. This second set of seven examinations may be roughly classified as follows: (8) analytic statics, dynamics of a particle, rigid dynamics, astronomy, hydrostatics, electricity, hydrodynamics; (9) advanced algebra (series, theory of numbers, theory of equations), symbolic operators, advanced analytics of two and of three dimensions; (10) analytic and spherical trigonometry, differential equations, definite integrals, plane curves; (11) analytic statics, dynamics, elasticity, hydrodynamics, electricity and magnetism; (12) theory of functions, elliptic functions, harmonic analysis (Bessel's functions, Fourier's functions, etc.); (13) a mixed paper, both pure and applied; (14) problem paper on (8) to (13). Here again, pure and applied mathematics are given equal weight.

About a third of the wranglers take the second part of the tripos at the end of their fourth year. In the first part a student is required to have a good general knowledge of the whole field of pure and applied mathematics, but in the second part there are four divisions of pure mathematics and four divisions of applied mathematics. The student chooses any two of these eight divisions for further specification. The questions embody the latest work in each division, and the student's rank is determined by the quality of his work in one direction. Sample papers are here added in illustration of the general character of the examination.*

PART I. SIXTH PAPER.

1. Find the conditions of equilibrium satisfied by a body which is wholly or partly immersed in a fluid, and free to turn about a fixed point.

A semi-circular lamina has one of the ends of its diameter smoothly hinged to a fixed point above the surface of a liquid, and floats with its plane vertical and its diameter half immersed. If the inclination of the diameter to the horizon is $\pi/4$, prove that the ratio of the density of the liquid to that of the lamina is $4(3\pi - 4) : 9\pi - 8$.

2. Find the center of pressure of a rectangle, completely immersed in a liquid, with one of its sides horizontal.

A rectangular block whose edges are of lengths $2a$, $2b$, $2c$ is divided by a plane through the center perpendicular to the edges of length $2c$, and the two halves are hinged together along edges parallel to those of length $2a$. The whole is then immersed in a liquid with the line of hinges inclined at an angle θ to the horizon and the dividing plane vertical, the hinges being in the upper face. Prove that the two halves will not separate unless

$$\left\{\left(1 - \frac{\sigma}{\rho}\right)c^2 - \tfrac{2}{3}b^2\right\} \cos\theta > 2bd,$$

where d is the depth of the center of gravity of the block, σ is the density of the block, and ρ that of the liquid.

3. A cylindrical diving-bell is lowered to a given depth below the surface of a liquid. Find the height to which the liquid rises inside the bell.

A cube of density ρ_0 floats on a liquid of density ρ, and a cylindrical diving-bell, of length l and cross section A, is placed over it and lowered until the cube just touches the top. Show that the height to which the liquid rises inside the bell is

$$[A(\rho - \sigma) - \rho a^2]\,[l(\rho - \sigma) - a(\rho - \rho_0)]\,/\,[\rho(\rho - \sigma)(A - a^2)],$$

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* We are permitted to reprint the questions by the courtesy of the Syndics of the Cambridge University Press, who hold the copyright. Complete sets of the tripos can be obtained from the publishers.

where σ is the density of the atmosphere, and a is the length of an edge of the cube.

4. Describe Smeaton's air-pump, and find the density in the receiver after n strokes.

A barometer tube, in the form of a flat-topped cylinder, is placed with its lower end in a vessel containing mercury, and the air is pumped out. Prove that, if x_n is the distance of the mercury from the top of the tube after n strokes,

$$(x_n - h)(ax_n + u) = (x_{n-1} - h)ax_{n-1},$$

where a is the cross section of the tube, u the volume of the barrel of the pump, and h the ultimate value of x_n.

5. Define the terms 'latent heat' and 'specific heat,' and show how the specific heat of a substance may be determined by observing the alteration in temperature of a given mass of water, into which a given mass of the substance at a given temperature is dropped.

A closed vessel of mass 40 grams and specific heat .095 is filled with 4.4 grams of a gas, the whole being at temperature 20° C. The vessel is then enclosed in a chamber through which steam at 100° C. is passed until the temperature becomes uniform, and the increase in mass is found to be .68 gram. Prove that the specific heat of the gas under the given conditions is .173, the latent heat of steam at 100° C. being 536.5.

6. Define *potential*, and find the force and the potential at any point due to a freely electrified spherical conductor.

A curve is drawn on the surface of a freely electrified oval conductor surrounded by air, dividing it into two parts on which the charges are E and E' respectively. The space, enclosed by the portion of the surface of the conductor on which the charge is E' and by the lines of force drawn from every point of the curve to infinity, is filled with a dielectric of S. I. C. K. Prove that the potential of the conductor is now diminished in the ratio $E + E' : E + KE'$.

7. Prove that the energy of a system of charged conductors is $\frac{1}{2}\Sigma EV$.

A condenser is formed of two parallel plates. Prove that, when the potentials of the plates are kept constant, the work done by the system in a small displacement is equal to the increase in the energy of the system.

8. Define the coefficients of potential of a system of conductors, and prove that $p_{rs} = p_{sr}$. Under what circumstances does $p_{rr} = p_{rs}$?

Three conductors A_1, A_2 and A_3 are such that A_3 is practically inside A_2. A_1 is alternately connected with A_2 and A_3 by means of a fine wire, the first contact being with A_3. A_1 has a charge E initially, A_2 and A_3 being uncharged. Prove that the charge on A_1, after it has been connected with A_3 n times, is

$$\frac{E\beta}{\alpha + \beta}\left\{1 + \frac{\alpha(\gamma - \beta)}{\beta(\alpha + \gamma)}\left(\frac{\alpha + \beta}{\alpha + \gamma}\right)^{n-1}\right\},$$

where α, β and γ stand for $p_{11} - p_{12}$, $p_{22} - p_{12}$ and $p_{33} - p_{12}$ respectively.

9. State and prove Kirchhoff's laws as to the distribution of electric currents in a system of linear conductors containing given electromotive forces.

A square $ABCD$ is formed of a uniform piece of wire, and the center is joined to the middle points of the sides by straight wires of the same material and cross section. A current is taken in at A and is drawn off at the middle point of BC. Prove that the equivalent resistance of the network between these points is $\frac{29}{48}$ of that of a side of the square.

10. Describe the tangent galvanometer.

A given current sent through the galvanometer deflects the magnet through an angle θ. The plane of the coil is slowly rotated round the vertical axis through the center of the magnet. Prove that, if $\theta > \frac{1}{4}\pi$, the magnet will describe complete revolutions, but if $\theta < \frac{1}{4}\pi$, the magnet will oscillate through an angle $\sin^{-1}(\tan\theta)$ on each side of the meridian.

PART I. TENTH PAPER.

1. If $\sum\limits_0^\infty u_n$ is a semi-convergent series of real terms, prove that the infinite product $\prod\limits_0^\infty (1 + u_n)$ tends towards a positive limit or towards zero according as $\sum\limits_0^\infty u_n^2$ is convergent or divergent.

If $u_0 = u_1 = u_2 = 0$ and when $n > 1$

$$u_{2n-1} = \frac{-1}{\sqrt{n}}, \qquad u_{2n} = \frac{1}{\sqrt{n}} + \frac{1}{n} + \frac{1}{n\sqrt{n}},$$

show that $\sum_0^\infty u_n$ and $\sum_0^\infty u_n^2$ are both divergent, but that $\prod_0^\infty (1 + u_n)$ is convergent.

2. Show that a rational integral congruence of degree n cannot have more than n roots incongruent to a prime modulus.

Prove that, if n is an odd prime,

$$x^{n-1} - 1 - [x^2 + 1(n-1)]\,[x^2 + 2(n-2)] \cdots \\ [x^2 + \tfrac{1}{2}(n-1)\tfrac{1}{2}(n+1)] \equiv 0 \pmod{n}$$

is an identical congruence.

What arithmetic theorems result from this?

3. Investigate the solution of the quartic $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$, and obtain the auxiliary cubic in the form $4a^3\theta^3 - Ia\theta + J = 0$.

If ϕ is the cross-ratio of the four roots, show that the correspondence between θ and ϕ may be written in the forms

$$(\phi+1)^2 I = 12a^2\theta^2(\phi^2 - \phi + 1), \quad (\phi+1)^3 J = 4a^3\theta^3(2\phi - 1)(\phi - 2).$$

4. Show that the hessian of a quantic is a covariant of that quantic.

If a ternary cubic is the product of three linear factors, prove that its hessian is the product of the cubic with a constant.

5. If u is a homogeneous function of $x_0, x_1, x_2, \cdots, x_m$ of degree n prove that

$$\left(x_0 \frac{\partial}{\partial x_0} + x_1 \frac{\partial}{\partial x_1} + \cdots + x_m \frac{\partial}{\partial x_m}\right) u = nu.$$

Let

$$\Omega \equiv x_0 \frac{\partial}{\partial x_1} + 2x_1 \frac{\partial}{\partial x_2} + 3x_2 \frac{\partial}{\partial x_3} + \cdots + mx_{m-1} \frac{\partial}{\partial x_m},$$

and

$$O \equiv mx_1 \frac{\partial}{\partial x_0} + (m-1)x_2 \frac{\partial}{\partial x_1} + (m-2)x_3 \frac{\partial}{\partial x_2} + \cdots + x_m \frac{\partial}{\partial x_{m-1}},$$

and let G be a homogeneous algebraic function of $x_0, x_1, \cdots, x_m$ of degree n such that the sum of the suffixes in each term is w.

Prove that $(\Omega O - O\Omega)G = (mn - 2w)G$.

6. If x, y, z are connected by a single equation, express

$$\frac{\partial z}{\partial x},\quad \frac{\partial z}{\partial y},\quad \frac{\partial^2 z}{\partial x^2},\quad \frac{\partial^2 z}{\partial x \partial y},\quad \text{and}\quad \frac{\partial^2 z}{\partial y^2}$$

in terms of differential coefficients of x with respect to y and z.

If

$$\frac{\partial(Z, z)}{\partial(x, y)} = 0, \quad \text{where} \quad \frac{\partial^2 z}{\partial x \partial y} - Z \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} = 0,$$

show that the same relations hold good on permuting the letters, that is to say

$$\frac{\partial(X, x)}{\partial(y, z)} = 0, \quad \text{where} \quad \frac{\partial^2 x}{\partial y \partial z} - X \frac{\partial x}{\partial y} \frac{\partial x}{\partial z} = 0, \quad \text{etc.}$$

7. Find the equation of a pair of tangents to $x^2/a^2+y^2/b^2=1$, the chord of contact being $\lambda x + \mu y + \nu = 0$.

Find the equation of a circle touching these two tangents and meeting the ellipse in the chord $x \cos \phi + y \sin \phi = p$, concurrent with the chords of contact; and if ϕ is the constant α, show that the locus of the intersection of the above-mentioned tangents is the conic

$$x^2/(a^2 \cos^2 \alpha) - y^2/(b^2 \sin^2 a) = (a^2 - b^2)/(a^2 \cos^2 a + b^2 \sin^2 a),$$

that is to say, an hyperbola confocal with a given ellipse.

8. Find the point equation of the circular points at infinity in any system of homogeneous coördinates, and deduce the corresponding tangential equation.

A parabola circumscribes a triangle, show that its axis touches a curve of the third class (a three-cusped hypocycloid) given by

$$\lambda^2 \frac{\partial \Omega}{\partial \lambda} + \mu^2 \frac{\partial \Omega}{\partial \mu} + \nu^2 \frac{\partial \Omega}{\partial \nu} = 0,$$

and the tangent at the vertex always touches the curve of the sixth class defined by the rationalized form of

$$\lambda^{\frac{1}{2}} \frac{\partial \Omega}{\partial \lambda} + \mu^{\frac{1}{2}} \frac{\partial \Omega}{\partial \mu} + \nu^{\frac{1}{2}} \frac{\partial \Omega}{\partial \nu} = 0,$$

where the line at infinity is $x + y + z = 0$ in point-coördinates, and the rational tangential equation of the circular points at infinity is $\Omega = 0$.

9. Find the equation of a normal at any point of the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1,$$

and deduce that normals meeting in (x_0, y_0, z_0) lie on the cone

$$\frac{x_0}{x-x_0}(b^2-c^2)+\frac{y_0}{y-y_0}(c^2-a^2)+\frac{z_0}{z-z_0}(a^2-b^2)=0.$$

If (x_0, y_0, z_0) lies on the ellipsoid and tangent planes be drawn to the ellipsoid at the five feet of normals other than (x_0, y_0, z_0), then the feet of the perpendiculars from the center on these tangent planes all lie in the tangent plane at $(-x_0, -y_0, -z_0)$, and are all situated on a rectangular hyperbola passing through the foot of the perpendicular from the center on the same plane.

10. Find the principal planes of the quadric

$$S \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy-1=0.$$

If $X \equiv ax+hy+gz$, $X' \equiv a'x+h'y+g'z$, etc., show that the set of conjugate diametral planes which $S=0$, $S'=0$ have in common is determined by the equation

$$\frac{\partial(S, S', S'')}{\partial(x, y, z)}=0,$$

where S'' is given by either $(A, B, C, F, G, H)(X', Y', Z')^2$ or $(A', B', C', F', G', H')(X, Y, Z)^2$.

PART II. DIVISIONS II, IV.

1. Prove that covariants of binary forms can be expressed in terms of symbolical factors of the type (ab) and a_x, and explain how to extend the result to ternary forms.

Prove that for binary forms the minimum weight of a perpetuant of degree four is seven, and explain how to extend the result to higher degrees.

2. A line l meets a conic S_1 in two conjugate imaginary points, P, Q. O is a point on a conic S_2 and OP, OQ meet S_2 in $P'Q'$; obtain a real construction for the line $P'Q'$.

One pair of opposite vertices of a quadrilateral are real and a second pair are conjugate and imaginary. Give a real construction for the third diagonal and the third pair of vertices.

State, in terms of cross ratios, the proposition that the angle at the center of a circle is double that at the circumference on the same arc, and give a direct proof of the statement.

3. Give an account of the properties of the twenty-seven

lines on a cubic surface, and explain how to deduce properties of the bitangents of a plane quartic curve.

Show that there exist sets of six bitangents which touch the same conic.

4. Prove the fundamental theorems relating to the parametric expressions for the coördinates of a point on a plane curve of deficiency 0 or 1.

In the case of a plane cubic find how many lines can be drawn such that, at each point where they meet the curve, a conic can be drawn having six-point contact with the curve.

Indicate briefly how the equation of any elliptic curve can be reduced to the form

$$y^2 = 4x^3 - g_2x - g_3.$$

VIRGIL SNYDER.

SHORTER NOTICES.

Ueber die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert. Bericht der Deutschen Mathematiker-Vereinigung, erstattet von MAX SIMON. Mit 28 Figuren im Text. Leipzig, B. G. Teubner, 1906. viii + 278 pp. Price, 8 marks.

Methodik der Elementaren Arithmetik in Verbindung mit Algebraischer Analysis. Von MAX SIMON. Mit 9 Textfiguren. Leipzig, B. G. Teubner, 1906. vi + 108 pp. Price, 3 marks 20 pf.

AMONG those occupying chairs of mathematics in the German universities there is no one who takes greater interest in the work of the secondary teacher than Professor Simon. He is earnest in his advocacy of reform, zealous in his application of the history of mathematics to the principles of teaching, and full of that good-humored argument that makes a man acceptable as a speaker in an assembly of teachers. Therefore it comes about that Professor Simon is able to command appreciative audiences for his addresses and a goodly circle of readers for his numerous literary efforts. It is rather in his work as a speaker, however, that he is most successful. His fund of enthusiasm, his genial countenance, and his action in address, all tend to lead his hearers to consider his arguments as wholes, without attending to minor inaccuracies of language or of statement. When, on the other hand, he attempts to put his ideas on paper, his larger thoughts, worthy though they may be, seem submerged in a sea of minor inaccuracies. In the case of the two books under review, the first is of such a nature as to invite even more errors than are usually found in Professor Simon's writings, and therefore, as may be inferred from what has been said, the reader will find much to vex his soul in looking over its pages.

In spite of these annoyances, however, each book has a distinct value, and each can be recommended to the teacher of mathematics, if not to the mere mathematician. The first book, "Ueber die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert," is taken from the Jahresbericht der Deutschen Mathematiker-Vereinigung, and was originally begun in the interests of the Encyklopädie der mathematischen Wissenschaften. Its scope is limited, as the title suggests, in time to the nineteenth century, and in subject to the geometry of the Gymnasium or Oberrealschule. Professor Simon begins with a general survey of the bibliography of the subject, particularly that of the history, the method of teaching, and the text-books relating to geometry. It is in such lists of books and articles, imperfect though they are, that the chief value of the work will be found to lie. The author then takes up a number of special topics as follows: The theory of parallels; the circle, including quadrature, inscriptibility of polygons, and the multisection of arcs; areas, including isoperimetry; the triangle, including a study of critical points and lines; polygons; plane configurations, including similarity, centers of gravity, and transversals; general space relations, in particular certain questions of stereometry, volumes and elementary surfaces, and spherics; special space relations, including the tetrahedron, polyhedron, and the Euler theorem; and trigonometry, plane and spherical.

Under each of these topics a brief historical survey is given, with a bibliography, and this is followed by a summary of the most important attempts at treating the subject in hand. Thus under the quadrature of the circle the historical survey is followed by a list (rather imperfect) of circle-squarers, a statement of approximate constructions, a few of the efforts at representing π numerically, and the history of a few such attempts as that of Hippocrates. In the same way the trisection problem, the 17-gon, the critical points of the triangle, and Euler's problem (with no reference to Hadamard's proof) are discussed. In no case is the treatment as thorough as in Klein's Vorträge, but it is always readable, and the bibliography opens up to the student a good part of the literature of the subject.

The "Methodik der elementaren Arithmetik" is a set of lectures delivered by Professor Simon in the summer semester of 1904. As such, they would naturally be expected to be more popular and less critical in details than the work already mentioned. They are simple in style, and were doubtless intended for teachers who felt the need of having their attention called to the modern treatment of certain fundamental ideas of mathematics. Hence we find a popular treatment of the idea of integral number, the operations with number and the fundamental laws involved, fractions, series, powers and roots, equations of the first three degrees, logarithms, and complex numbers. It is such a course of lectures as might profitably be attended by the high school teacher in this country, and the book should find a considerable circle of readers here as well as in Germany.

While the books have this value, however, it must be said that the array of inaccuracies which confronts the reader seems quite inexcusable. The number of the errata at the end of the Entwicklung, upwards of 135, might easily be doubled from a single reading. Moreover there are other and more serious points of criticism. Professor Simon's dismissal of Dr. H. Schotten's "Inhalt und Methode" by the simple word "schwächer" is unwarranted, for the book contains a good deal that is of value to the teacher of geometry. His statement that Rupert's little book on famous propositions of geometry seems to be merely an extract from Ball cannot be justified; if the book were to be attacked it should be on other grounds. Moreover Professor Simon's knowledge of the relative values of books, monographs, and proofs of a non-German character is limited, as the merest glance at his list of text-books will show. An example of this is seen in his inclusion of an edition of Sacrobosco's Sphæra in a list of works on modern spherics. His statement (Methodik, p. 16) that in Sargon's time the position system was known is liable to serious misconstruction. His statement that the French call the multiplication table the "table de Pythagore" will also be misunderstood, for the name was a common one in the works of the mediæval writers. That Gemma Frisius said "unitas ipsa non est numerus," in 1576

(Methodik, page 8), is impossible, for he was dead then. Moreover it was not at all a new idea when he published his arithmetic in 1540, for it had been generally repeated by writers from the time of Boethius down. His statement that the study of the Pascal triangle began with Stifel is also incorrect, for it was apparently known long before Stifel's works were written, appearing for example in the engraved title page of "Eyn Newe Vnnd wolgegründte vnderweysung aller Kauffmanss Rechnung" published by Apianus at Ingolstadt in 1527.

DAVID EUGENE SMITH.

Elements of Descriptive Geometry. By O. E. RANDALL. Boston, Ginn & Company, 1905. iv + 209 pp.

Elements of Descriptive Geometry. By CHARLES E. FERRIS. New York, American Book Company, 1905. vii + 127 pp.

THESE books meet the demand for texts in which the objects are shown in the figures in the third quadrant. They follow the method of Warren and Church in presenting first the analysis and then the construction of a given problem. The figures are not in a separate volume or collection of plates, but in the text where most convenient for the reader.

The authors present the fundamental ideas with care, and Dr. Randall supplements the figures for the first fifty pages with "pictorial drawings" showing the projection planes and the object as they actually appear to the eye of the observer.

It will be noticed that the chapters on "Lines and surfaces" retain the definitions and principles used by the older American writers. If it is necessary to set up for engineering students a complete theory of curves and surfaces without analysis, it is desirable that in doing it naïve expressions that may confuse students of mathematics should be avoided.* For example, in Dr. Randall's book, § 203, we read * * * "The portion of the line generated by the point while moving from one position to its consecutive position is called an *elementary line* and while in theory it may be regarded as having length, practically speaking it has none."

The error of classifying a warped surface as a surface of single curvature is implied by Dr. Randall in § 221, where we find:

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*Compare Rohn, Lehrbuch der darstellenden Geometrie, Leipzig, 1906, p. 219.

"Depending upon the character of the generatrix we have two classes of surfaces: first those which are generated by straight lines or those which have rectilinear elements; and second those which are generated by curved lines known as *surfaces of double curvature*."* Professor Ferris also states in Chapter IV on "Double curved surfaces" that "double curved surfaces have no right line elements" and he classes warped surfaces by themselves.

Monge's original sharp classification of surfaces into planes, single curved or *developable* surfaces, and double curved surfaces† has been mangled by several American writers including Warren‡ and Church.§ The present classification by Dr. Randall into single curved surfaces, § 222; surfaces of revolution, § 223; and double curved surfaces of revolution, § 224; seems without adequate foundation.

Professor Ferris has a brief chapter on shades and shadows and one on perspective.

L. I. HEWES.

NOTES.

THE Thirteenth Annual Meeting of the AMERICAN MATHEMATICAL SOCIETY will be held in New York on Friday and Saturday, December 28–29. Friday morning will be devoted to a joint session with Section A of the American association for the advancement of science and the Astronomical and astrophysical society of America. President OSGOOD's address will probably be delivered at the opening of the Friday afternoon session. The Council will meet on Friday afternoon. The annual election of officers and other members of the Council will be held on Saturday morning. Special railroad rates will be available for this meeting.

THE concluding (October) number of volume 7 of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "Weierstrass' theorem and Kneser's theorem on transversals for the most general case of an extremum of a simple definite integral," by O. BOLZA; "Area of curved surfaces," by J. PIERPONT; "On multiply transitive groups," by W. A. MANNING; "Zur Theorie der vollständig reduciblen Gruppen, die zu einer Gruppe linearer homogener Substitutionen gehören," by L. STICKELBERGER; "On commutative linear algebras in which division is always uniquely possible," by L. E. DICKSON; "On the order of linear homogeneous groups," by H. F. BLICHFELDT; "On automorphic groups whose coefficients are integers in a quadratic field," by J. I. HUTCHINSON; "A class of periodic solutions of the problem of three bodies with applications to the lunar theory," by F. R. MOULTON; "A problem in the reduction of hyperelliptic integrals," by J. H. MCDONALD; "On the differential invariants of a plane," by C. N. HASKINS. Notes and errata, volumes 6, 7.

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* See also Randall, §§ 223, 224.

† G. Monge, Géométrie descriptive. Paris, 1858, p. 120.

‡ S. Edward Warren, Descriptive geometry. New York, 1860, p. 4 and p. 194.

§ A. E. Church, Descriptive geometry and shades and shadows. New York, 1870, p. 41.

Professor W. F. OSGOOD has retired from the Editorial Committee of the *Transactions*, and is succeeded by Professor H. S. WHITE.

THE opening (October) number of volume 8 of the *Annals of Mathematics* contains the following papers: "The fundamental laws of addition and multiplication in elementary algebra," by E. V. HUNTINGTON; "On a criterion of Pringsheim's for expansibility in Taylor's series," by M. B. PORTER; "Multiply perfect numbers of three different primes," by R. D. CARMICHAEL.

AT the Stuttgart meeting of the Deutsche Mathematiker-Vereinigung, Professor A. VON BRILL, of the University of Tübingen, was elected president; Professors M. KRAUSE, of Dresden, and A. SCHOENFLIES, of Königsberg, were elected to the Council. The next meeting will be held at Dresden during the summer of 1907. The society now numbers 688 active members.

THE Cambridge University Press announces that the following works will soon appear: Theory of differential equations, volume 5, by A. R. FORSYTH; Theoretic mechanics, an introductory treatise on the principles of dynamics, with applications and numerous examples, by A. E. H. LOVE.

THE following parts of the Encyklopädie der mathematischen Wissenschaften are announced by the firm of B. G. Teubner, Leipzig, to be in the press, and will probably appear in December: III, 1, Analysis situs, by P. HEEGARD and M. DEHN; V, 1, Kinetische Theorie der Materie, by L. BOLTZMANN and J. NABL; Kapillarität, by H. MINKOWSKI; V, 2, Beziehungen zwischen electrostatischen und magnetischen Zustandsänderungen einerseits und elastischen und thermischen andererseits, by F. POCKELS. Also those mentioned in the BULLETIN, volume 12, page 465, except III, 2, Abzählende Methoden, by H. G. ZEUTHEN; and Allgemeine Theorie der höheren ebenen algebraischen Kurven, by L. BERZOLARI, which have already appeared.

Of the French edition of the encyclopedia there have thus far appeared the first sections of volumes I 1, 3 and 4.

THE following Americans have received the doctorate from European universities, with mathematics as the major subject, during the six semesters ending August 1, 1906: J. W. BRADSHAW (Strassburg, 1904), "Ueber die Flächendichtigkeit der Elektricität auf unendlich langen Cylindern;" J. L. COOLIDGE (Bonn, 1904), "The dual projective geometry of elliptic and spherical space"; F. J. DOHMEN (Greifswald, 1904), "Darstellung der Berührungstransformationen in Konnexcoordinaten"; D. C. GILLESPIE (Göttingen, 1906), "Anwendungen des Unabhängigkeitsatzes auf die Lösung der Differentialgleichung der Variationsrechnung"; A. L. P. WERNICKE (Göttingen, 1904), "Ueber die Analysis Situs mehrdimensionaler Räume"; W. D. A. WESTFALL (Göttingen, 1905), "Zur Theorie der Integralgleichungen."

To the list of American doctorates for 1906 having mathematics for the major subject should be added the names of W. R. LONGLEY (Chicago), "A class of periodic orbits of an infinitesimal body subject to the attraction of n finite bodies;" and F. L. GRIFFIN (Chicago), "Certain periodic orbits of k finite bodies revolving about a relatively large central mass."

UNIVERSITY OF PARIS.—The following courses in mathematics are given during the present semester:—By Professor G. DARBOUX: Systems of curvilinear coördinates, two hours.—By Professor E. GOURSAT: Elements of the theory of analytic functions, two hours; Conference, one hour.—By Professor P. PAINLEVÉ: General laws of motion and equilibrium, two hours.—By Professor L. RAFFY: Applications of analysis to geometry, one hour; General mathematics, two hours.—By Professor H. POINCARÉ: Limitations of the newtonian law, two hours.—By Professor J. BOUSSINESQ: Cooling of liquids, two hours. — By Professor G. KOENIGS: Dynamics of machines, two hours. — By Professor E. BOREL: Theory of functions, one hour. — By Professor J. HADAMARD: General mathematics, two hours; Conference, four hours. — By Dr. P. PUISEUX: Conference on mechanics, two hours. — By Dr. E. BLUTEL: Conference on algebra, two hours. In the École Normale. — By Professor J. TANNERY: Calculus, two hours, and the courses mentioned above, by Professors Hadamard, Raffy and Borel.

PROFESSOR J. MOLK, of the University of Nancy, and Professor T. LEVI-CIVITA, of the University of Padua, have been elected to membersip in academy of sciences at Halle.

PROFESSOR G. ROST, of the University of Würzburg, has been promoted to a full professorship of mathematics.

PROFESSOR W. O. WIEN, of the University of Würzburg, has declined the invitation of the University of Berlin to be professor of theoretical physics, as successor to the late Professor Drude.

DR. M. BOHREN has been appointed docent in mathematics and theory of probabilities at the University of Bern.

PROFESSOR G. LANDSBERG, of the University of Breslau, has been appointed associate professor of mathematics at the University of Kiel.

PROFESSOR TH. SCHMID, of the technical school at Vienna, has been promoted to a full professorship of descriptive geometry.

PROFESSOR K. ZSIGMONDY, of the technical school at Prague, has been appointed professor of mathematics at the technical school of Vienna.

DR. H. REISSNER, of the technical school at Charlottenburg, has been appointed professor of mathematics at the technical school at Aachen.

AT the University College of London, Mr. N. G. DUNBAR has been appointed demonstrator in applied mathematics.

At the University of Michigan, Dr. J. W. GLOVER has been promoted to a junior professorship of mathematics and insurance. The insurance courses have been broadened by the addition of a seminary for advanced students.

AT Lehigh University, Mr. J. H. OGBURN has been appointed assistant professor of mathematics.

AT George Washington University, Mr. P. N. PECK has been promoted to an assistant professorship of mathematics, and Mr. G. A. ROSS has been appointed instructor in mathematics.

AT the College of the City of New York, Mr. S. J. MAGARGE has been appointed tutor in mathematics. Mr. R. F. DEIMEL has been transferred to the department of physics.

THE following academic appointments are also announced: DR. F. L. GRIFFIN as instructor in mathematics at Williams College, Williamstown, Mass.; Mr. R. M. BARTON and Mr. F. C. MOORE as instructors in mathematics at Dartmouth College; Mr. P. DORWEILAR and Mr. C. F. HAGENOW as instructors in mathematics at the Armour Institute of Technology; Mr. F. F. DECKER as instructor in mathematics at Syracuse University; Mr. L. WEEKS as instructor in mathematics at Purdue University; Mr. W. M. SMITH as tutor and Mr. J. W. COLLITON as instructor in mathematics at Lafayette College; Mr. W. S. PEMBERTON and Mr. P. F. HEDGES as assistants in mathematics at the University of Missouri.

PROFESSOR ERNESTO CESÀRO, of the University of Naples, died September 12, at the age of forty-seven years.

CATALOGUES of second-hand mathematical books: C. Winter, Dresden, Galleriestrasse 8, catalogue no. 97, 287 titles in pure and applied mathematics. — H. Sotheran and Co., London, 140 Strand, catalogue 666, 1437 titles in mathematics and sciences. — Carlo Clausen, Turin, Via Po 11, catalogue no. 128, 678 titles in mathematics, physics, and astronomy. — Karl W. Hiersemann, Leipzig, Königstrasse 3, Katalog 304, a few 16th–18th century works.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BIBLIOGRAPHY of Russian literature upon mathematics, pure and applied, and upon the natural sciences, for 1902. Edited by the Society of naturalists of Kiev. Kiev, 1905. 8vo. (Russian.)

BORELIUS (J.). Zur Theorie der Filarevoluten und Parallelkurven ebener und sphärischer Kurven. Lund, 1906. 8vo. 63 pp. M. 2.00

DEPÈNE. Ueber die Flächen gewisser einem Dreieck eingeschriebener Dreiecke. Breslau, 1906. 4to. 13 pp. M. 1.50

FATTOR (L.). Di un metodo per determinare le regole di divisibilità relative ad un numero primo con 10. Cividale, Stagni, 1906. 8vo. 7 pp

——. Una regola pratica per avere il periodo di $1/p$ essendo p primo con 10. Cividale, Stagni, 1906. 8vo. 4 pp.

GALILEI (G.). Le opere. Edizione nazionale sotto gli auspici di Sua Maestà il Re d'Italia. Vol. XVIII. Firenze, Alfani, 1906. 4to. 545 pp.

GEPP (H.). Ueber Inversionssummen. (Diss.) Giessen, 1906. 8vo. 40 pp.

HAWLITSCHEK (K.). Integration irrationaler Differentiale. (Progr.) Prag-Neustadt, 1906. 8vo. 9 pp.

HEYDEN (A. F. V.). First steps in the calculus. London, Arnold, 1906. 8vo. 224 pp. Cloth. 3s.

JAHRBUCH uber die Fortschritte der Mathematik. Begrundet von C. Ohrtmann. Unter Mitwirkung von F. Müller und A. Wangerin sowie der Berliner Mathematischen Gesellschaft herausgegeben von E. Lampe. Vol. 35. Heft 1. Berlin, 1906. 8vo. 496 pp. M. 17.80

KOERBER (F.). Transformator fur sphärische Coordinaten. Berlin, Reimer, 1906. 8vo. M. 1.50

LANNER (A.). Die neueren Darstellungen der Grundprobleme der reinen Mathematik. Berlin, Salle, 1906. 8vo. M. 3.00

LEBESGUE (H.). Leçons sur les séries trigonométriques. Paris, Gauthier-Villars, 1906. 8vo. 7 + 128 pp. F. 3.50

McCORMACK (T. J.). See MACH (E.).

MACH (E.). Space and geometry in the light of physiological, psychological and physical inquiry; from the German by T. J. McCormack. Chicago, Open Court, 1906. 12mo. 2 + 148 pp. Cloth. $1.00

MANSION (P.). Précis de la théorie des fonctions hyperboliques, suivi d'une théorie purement analytique des fonctions circulaires. 2e édition, revue et augmentée. Paris, Gauthier-Villars, 1906. 8vo. 44 pp. F. 1.00

MEYER (P.). Beweis eines von Euler entdeckten Satzes, betreffend die Bestimmung von Primzahlen. Strassburg, 1905. 8vo. 29 pp. M. 1.50

PETIT BOIS (G.). Tables d'intégrales indéfinies. Liége, 1906. 4to. 12 + 154 pp. M. 8.50

PIEL (C.). Ueber die Kegelschnitte, welche durch 3 Punkte und 2 Tangenten oder durch 2 Punkte und 3 Tangenten bestimmt sind, und die Kegelschnittsysteme ($3\,p$, $1\,l$) und ($1\,p$, $3\,l$). Strassburg, 1905. 8vo. 73 pp. M. 2.00

Poincaré (H.). El valor de la ciencia. Versión española de Emilio González Llana. Madrid, Gutenberg, 1906. 264 pp.

Pucciano (G.). Indagine sul potenziale logaritmico e newtoniano. Messina, d'Amico, 1906. 8vo. 38 pp.

Rantsov. Pure mathematics. With a popular exposition of the foundations of higher analysis. St. Petersburg, 1906. 8vo. (Russian.) M. 3.60

Roitman (D.). Mathematics as a science and as a subject of general culture. St. Petersburg, 1906. 8vo. 56 pp. (Russian.) R. 0.50

Steingräber (W.). Ueber partielle Differentialgleichungen erster Ordnung im R_4. Greifswald, 1906. 8vo. 46 pp. M. 1.50

Tognelli (G.). Una applicazione della teoria dei residui di variabile complessa. Livorno, Belforte, 1906. 8vo. 49 pp.

II. ELEMENTARY MATHEMATICS.

Bieler (A.). Arithmetisches Lehr- und Uebungsbuch für Knaben-Mittelschulen. Teil I: Bis zu den Gleichungen zweiten Grades mit mehreren Unbekannten einschliesslich. Teil II: Reihenlehre, Zinseszins-Rechnung und Anfangsgründe der Trigonometrie. Leipzig, Teubner, 1906. 8vo. 323 pp. Cloth. M. 2.00

Ceramicola (F.). Multipli, summultipli, rapporti e proporzioni di grandezze omogenee commensurabili. Arezzo, Sinatti, 1906. 8vo. 33 pp.

Child (J. M.). See Lock (J. B.).

Cracknell (A. G.). See Workman (W. P.).

Deakin (R.). New matriculation algebra. With a selection of graphs. 4th edition. London, Clive, 1906. 8vo. 512 pp. Cloth. (University tutorial series.) 3s. 6d.

Edwardes (E. J.). Elements of plane geometry. London, Arnold, 1906. 8vo. 226 pp. Cloth. 3s. 6d.

F. I. C. Eléments d'algèbre, avec de nombreux exercices. Paris, Poussielgue, 1906. 16mo. 8 + 413 pp.

Feist (A.). Mathematische Aufgaben. Verzeichnis der mathematischen Aufgaben, die bei den Reifeprüfungen der 9-stufigen höheren Lehranstalten des Herzogtums Braunschweig von Ostern 1892 bis Ostern 1903 bearbeitet worden sind. Braunschweig, 1906. 4to. 35 pp. M. 1.50

Géométrie. Cours supérieur; par une réunion de professeurs. Paris, Poussielgue, 1906. 16mo. 8 + 323 pp.

Hellgren (A. E.). Allmánna metoder vid plangeometriska teorems och problems lósning. Part 1: 3d edition. Part II: 2d edition. Stockholm, 1906. 8vo. 239 pp. Cloth. M. 7.00

Hopfner (F.). Beitrag zur konstruktiven Lösung planimetrischer Aufgaben. (Progr.) Prag-Neustadt, 1906. 8vo. 9 pp.

Hoyos y Juliá. Nociones y ejercicios de aritmética y geometria. Zamora, 1906. 159 pp.

Lock (J. B.) and Child (J. M.). Trigonometry for beginners. London, Macmillan, 1906. 8vo. 204 pp. Cloth. 2s. 6d.

Maisón (C. C.). Trigonometry simplified. A practical treatise on the subject of trigonometry. Cleveland, 1906. 12mo. 61 pp. $0.50

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THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and thirtieth regular meeting of the Society was held in New York City on Saturday, October 27, 1906, the following thirty-three members being in attendance:

Professor G. A. Bliss, Professor E. W. Brown, Dr. W. H. Bussey, Professor F. N. Cole, Miss E. B. Cowley, Dr. W. S. Dennett, Professor T. S. Fiske, Miss Ida Griffiths, Mr. S. A. Joffe, Professor Edward Kasner, Professor E. O. Lovett, Professor T. E. McKinney, Professor James Maclay, Professor Max Mason, Professor Richard Morris, Professor W. F. Osgood, Professor James Pierpont, Mr. H. W. Reddick, Professor L. W. Reid, Dr. R. G. D. Richardson, Miss S. F. Richardson, Miss I. M. Schottenfels, Professor Charlotte A. Scott, Mr. L. P. Siceloff, Professor D. E. Smith, Professor P. F. Smith, Professor J. H. Tanner, Professor Oswald Veblen, Mr. H. E. Webb, Professor H. S. White, Miss E. C. Williams, Professor J. E. Wright, Professor J. W. Young.

President W. F. Osgood presided at the morning session, Ex-President T. S. Fiske at the afternoon session. The Council announced the election of the following persons to membership in the Society: Professor A. F. Carpenter, Hastings College; Dr. H. M. Dadourian, Yale University; Mr. T. E. Gravatt, Pennsylvania State College; Rev. A. S. Hawkesworth, Allegheny, Pa.; Mr. H. R. Higley, Pennsylvania State College; Dr. Mario Kiseljak, Fiume, Hungary; Dr. Emanuel Lasker, New York, N. Y.; Professor Ernest Lebon, Lycée Charlemagne, Paris; Dr. R. L. Moore, Princeton University; Mr. W. P. Russell, Pomona College; Professor J. H. Scarborough, State Normal School, Warrensburg, Mo.; Mr. L. P. Siceloff, Columbia University; Professor Cyparissos Stephanos, University of Athens. One application for membership in the Society was received.

A list of nominations of officers and other members of the Council was adopted and ordered placed on the official ballot for the annual election at the December meeting. A committee was appointed to audit the Treasurer's accounts for the current year.

Professor W. F. Osgood tendered his resignation from the Editorial Committee of the *Transactions*, finding it impossible to assume the burdens of the office. The vacancy was filled by the appointment of Professor H. S. White.

The following papers were read at this meeting:

(1) Miss S. F. RICHARDSON: "Note on poristic systems of polygons."

(2) Professor R. D. CARMICHAEL: "Multiply perfect numbers of four different primes."

(3) Dr. ARTHUR RANUM: "On Jordan's linear congruence groups."

(4) Professor BEPPO LEVI. "Geometrie projettive di congruenza e geometrie projettive finite."

(5) Professor CHARLOTTE A. SCOTT: "Note on regular polygons."

(6) Professors MAX MASON and G. A. BLISS: "Some problems of the calculus of variations in space with variable end points."

(7) Professor EDWARD KASNER: "Note on the transformations of dynamics."

(8) Professor G. A. MILLER: "Groups of order p^m containing exactly $p + 1$ abelian subgroups of order p^{m-1}."

(9) Professor G. A. MILLER: "The groups in which every subgroup is either abelian or hamiltonian."

Professor Levi's paper was communicated to the Society through Professor Osgood. In the absence of the authors, the papers of Professor Carmichael, Dr. Ranum, Professor Levi, and Professor Miller were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Miss Richardson's paper appears in full in the present number of the BULLETIN.

2. In this paper Professor Carmichael shows that there are but two multiply perfect numbers of (only) four different primes, and that these are $2^5 \cdot 3^3 \cdot 5 \cdot 7$, of multiplicity 4; and $2^9 . 3 \cdot 11 \cdot 31$, of multiplicity 3.

3. Dr. Ranum proves that if G is any Jordan group, mod p^a, p being prime, and if G_i $(i = 0, 1, \cdots, a)$ is the invariant subgroup whose matrices are congruent to identity, mod p^i, then, except for certain low values of p, every matrix in G of

period $p^i (i = 0, \cdots, a - 1)$ is contained in M_{a-i}, but not in M_{a-i+1}, and conversely; moreover every matrix of period p^i $(i = 0, \cdots, a)$ is commutative with every matrix of period p^{a-i}. These properties of Jordan groups are derived for the sake of their application to the more general linear congruence groups given in a paper entitled "The group of classes of congruent matrices, etc.," presented at the summer meeting of September, 1906. The great majority of the latter groups are thus shown to be not only new linear groups, but abstractly new, *i. e.*, they are not simply isomorphic with any Jordan groups of the same degree, or subgroups thereof.

4. In a note published in the *Transactions* (April, 1906) Drs. Veblen and Bussey have characterized the spaces consisting of a finite number of points in which a projective geometry exists. Professor Levi recalls that he has already occupied himself with spaces which include these as a particular case, in a paper on the "Fondamenti della metrica projettiva" (*Memorie* of the Turin Academy, series 2, volume 54, 1904). The coördinates of the points of these spaces are classes of numbers and rational expressions with rational (or entire) coefficients of a certain system (finite or not) of symbols, corresidual according to a determined irreducible modulus M. In these spaces a projective geometry exists which the author calls one of congruence.

Professor Levi indicates further how for such geometries the theorem of von Staudt is changed. After showing that, in any projective geometry, a transformation ϕ which converts harmonic groups into harmonic groups and leaves fixed the points $0, 1/0, 1$ is characterized by the equality

$$\phi[f(\xi, \eta, \zeta, \cdots)] = f[\phi(\xi), \phi(\eta), \phi(\zeta), \cdots],$$

where f is a rational function, he demonstrates that, if the number of the points of a right line is finite and the modulus M does not contain numbers, the study of the transformation ϕ corresponds to the study of the birational transformations with rational coefficients of the variety which represents the modulus M in itself. If, on the contrary, to the modulus M belongs a prime number p and the question is precisely about finite geometries in which the abscissas of the points of a right line belong to a Galois field $G(p^n)$, instead of the theorem of Staudt the following can be enunciated: Determine any succession whatever $n', n'', n''', \cdots, n$ of divisors of n, with $n' > 1$, and let each of them be equal to the preceding multiplied by a prime. To this succession of numbers $n^{(k)}$ one can get to correspond a succession of Galois fields $G(p^{n^{(k)}})$ contained in $G(p^n)$ and each containing the preceding, and a series of transformations ϕ characterized by the fact that each of them leaves fixed the points whose abscissas belong to one of these fields $G(p^{n^{(k)}})$, but removes the points of the next. The points belonging to one of these fields remain fixed as soon as one of them, not belonging to the preceding, remains fixed. To make sure that a ϕ comes to the identity it is necessary to verify at the most that it has as many fixed points, each rationally independent of those formerly considered, as there are prime divisors of the number n.

5. Miss Scott gave constructions for the regular pentagon, heptagon and nonagon by means of rectangular hyperbolas cutting a given circle at four vertices of the desired polygons; and from these deduced constructions for four vertices of the polygon on a given rectangular hyperbola.

6. The problem of finding a curve $y = y(x)$, $z = z(x)$, joining two given fixed points and minimizing an integral

$$I = \int f\left(x, y, z, \frac{dy}{dx}, \frac{dz}{dx}\right) dx$$

has been discussed according to recent methods by several writers, but comparatively little seems to have been done on the problem in which the end points are allowed to vary on surfaces or curves. In the paper of Professors Mason and Bliss the necessary conditions for a minimum when the end points are variable, analogous to those in the plane, and also sufficient conditions, have been derived. The problem in space has peculiarities essentially different from those of the corresponding problem in the plane on account of the fact that the end points may vary not only on curves but also on surfaces. The theory of congruences of curves plays an important part because the family of extremals transversal to a curve or surface is itself a congruence. A feature of the paper is the use of the parameter representation.

7. Appell has shown that the system of trajectories of a particle moving in a (positional) field of force is converted by any projective transformation into the system of trajectories corresponding to some other field of force. Professor Kasner showed, synthetically and analytically, that no other contact transformations possess this property.

8. Professor Miller's first paper appears in full in the present number of the BULLETIN.

9. Professor Miller's second paper is devoted to an extension of the results in the paper on "Non-abelian groups in which every subgroup is abelian" published in volume 4 of the *Transactions*. The principal theorems may be stated as follows: There is one and only one group of order 2^m which involves operators whose orders exceed four and satisfies the additional conditions that every subgroup is either abelian or hamiltonian and that at least one subgroup is hamiltonian. If every subgroup of a group of order 2^m, $m > 4$, is either abelian or hamiltonian and if it contains at least one hamiltonian subgroup, the entire group is hamiltonian. If a group contains at least one hamiltonian subgroup and if all its subgroups are either abelian or hamiltonian, it is the direct product of the hamiltonian group of order 2^m and an abelian group of odd order, unless it is the group of order 24 which does not contain a subgroup of order 12. There are only two non-hamiltonian groups which contain at least one hamiltonian subgroup and whose other subgroups are either abelian or hamiltonian.

F. N. COLE,
Secretary.

THE STUTTGART MEETING OF THE DEUTSCHE MATHEMATIKER-VEREINIGUNG.

THE annual meeting of the Deutsche Mathematiker-Vereinigung was held at Stuttgart September 17–20, 1906, forming a section of the seventy-eighth convention of the Deutsche Naturforscher und Aerzte. The following papers were presented:

(1) O. BLUMENTHAL, Aachen: "Integral transcendental functions and Picard's theorem" (report).

(2) A. PRINGSHEIM, Munich: "Fourier's integral theorem."

(3) G. Faber, Karlsruhe: "Series of Legendre's polynomials."

(4) O. Perron, Munich: "Singular points on the circle of convergence."

(5) F. Hartogs, Munich: "Recent investigations of analytic functions of several variables" (report).

(6) P. Stäckel, Hanover: "Power series in several variables."

(7) D. Hilbert, Göttingen: "The nature and purposes of the theory of integral equations."

(8) E. Hilb, Erlangen: "An extension of Klein's oscillation theorem."

(9) M. Krause, Dresden: "On functions of real variables."

(10) P. Koebe, Göttingen: "On the conformal mapping of multiply connected plane regions."

(11) F. Meyer, Koenigsberg: "The application of the generalized euclidean algorithm to the formation of resultants."

(12) P. Schafheitlin, Berlin: "On the theory of Bessel's functions."

(13) A. Schoenfliess, Königsberg: "Report on the theory of point manifoldnesses."

(14) G. Hessenberg, Grunewald: "Involution for transfinite ordinal numbers."

(15) G. Landsberg, Breslau: "On total curvature."

(16) K. Rohn, Leipzig: "Linear construction for cubic curves."

(17) Th. Schmid, Vienna: "Constructive treatment of the axis complex."

(18) H. Wiener, Darmstadt: "Models for twisted curves of the third order."

(19) C. Juel, Copenhagen: "Non-analytic curves in three dimensions."

(20) R. Müller, Brunswick: "Determination of the pole in motion of a plane system at branching points."

(21) C. Runge, Göttingen: "Graphical solutions of differential equations."

(22) R. Mehmke, Stuttgart: "New mechanisms for solution of dynamical problems."

(23) A. Wagenmann, Stuttgart: "Mathematical theory of evolution."

1. Professor Blumenthal's paper was devoted to functions of infinite order. Reference was made to a new generalization of

Picard's theorem by Rémoundos (Thèse, Paris, 1906). The speaker's results are to appear in a volume of the Collection de monographies, dirigée par Borel.

2. In comparison with the thorough investigations of the limits within which Fourier's series is valid, the treatment of Fourier's integral seems somewhat inadequate. In addition to integrability and conditions which are necessary for the series, the integral theorem requires also that the function satisfy certain complicated conditions at infinity. Professor Pringsheim undertakes to replace the latter by simple sufficient conditions by carrying over Dirichlet's condition (generalized by DuBois-Reymond and C. Jordan) to intervals of the form $(A, +\infty)$, $(-A, -\infty)$. For this purpose he designates as quasi-monotone a function which is single-valued and finite in an interval (A, B) and can be expressed as the sum of two finite, monotone parts. The condition is question may then be formulated: At infinity (*i. e.*, for $x > A$ and $x < -A$ where A is an arbitrary positive quantity) $f(x)$ must converge towards zero monotone or quasi-monotone. The proof is to appear in the *Annalen*, also a more detailed account in the *Jahresbericht*.

3. Dr. Faber proved the following proposition: Denoting Legendre's polynomials by $P_\nu(x)$, the singular points β of the function defined by the series $\sum_{\nu=0}^{\infty} a_\nu P_\nu(x)$ are connected with the singular points of the function $\sum_{\nu=0}^{\infty} a_\nu x^\nu$ by the relation $\beta = \frac{1}{2}(\alpha + \alpha^{-1})$.

4. Dr. Perron generalized the following two propositions:

I. If a function $\sum a_n x^n$ admits on the circle of convergence only one singular point ρ, and if this is a rational pole, then a_n / a_{n+1} approaches ρ as a limit when n becomes infinite (Vivanti-Gutzmer, Theorie der eindeutigen analytischen Funktionen, page 397).

II. If there exists a limit $\rho = \lim_{n=\infty} (a_n / a_{n+1})$, then ρ is a singular point, although not necessarily a rational pole nor the only singular point on the circle of convergence. (Hadamard: La série de Taylor, page 25.)

Theorems I and II admit the following generalizations:

I*a*. If the singular points on the circle of convergence are all rational poles $\rho_1, \rho_2, \cdots, \rho_k$ of order $m_1, m_2, \cdots, m_k$ respectively and the greatest m occurs p times, then there exist p numbers $\xi_1, \xi_2, \cdots, \xi_p$ such that

$$\lim \frac{a_n + \xi_1 a_{n+1} + \cdots + \xi_p a_{n+p}}{max\,|\,a_n, a_{n+1}, \cdots, a_{n+p}\,|} = 0$$

and the roots of the equation $\rho^p + \xi_1\rho^{p-1} + \cdots + \xi_p = 0$ are precisely the p poles with the highest degree of multiplicity.

I*b*. With the hypotheses of I*a*, if $\rho^{p-1} + \eta_1\rho^{p-2} + \cdots + \eta_{p-1}$ is a divisor of $\rho^p + \xi_1\rho^{p-1} + \cdots + \xi_p$, the quotient

$$\frac{|\,a_n + \eta_1 a_{n+1} + \cdots + \eta_{p-1}a_{n+p-1}\,|}{max\,|\,a_n, a_{n+1}, \cdots, a_{n+p-1}\,|}$$

after a fixed value of n remains greater than a certain positive number σ.

II*a*. If p is the least number for which there exists a set of numbers $\xi_1, \xi_2, \cdots, \xi_p$ as in I*a*, and if, $\rho^{p-1} + \eta_1\rho^{p-2} + \cdots + \eta_{p-1}$ being a divisor of $\rho^p + \xi_1\rho^{p-1} + \cdots + \xi_p$, the quotients

$$\frac{|\,a_n + \eta_1 a_{n+1} + \cdots + \eta_{p-1}a_{n+p-1}\,|}{max\,|\,a_n, a_{n+1}, \cdots, a_{n+p-1}\,|}$$

after a certain value of n remain greater than a positive σ, then (1) the roots of the equation $\rho^p + \xi_1\rho^{p-1} + \cdots + \xi_p = 0$ all have the same value; (2) this absolute value is equal to the radius of of convergence; (3) the roots are singular points.

5. In connection with Weierstrass's lemma and the proofs by Poincaré and Cousin that a single-valued function meromorphic in the finite plane can be expressed as the quotient of two integral functions, Dr. Hartogs referred to a paper by Hahn in the *Wiener Monatshefte* in which the investigation of the zero-manifoldnesses of analytic functions of several variables is completely worked out and the problem of resolving integral functions into prime factors is solved. The investigation of those manifoldnesses which consist of the singular points of an analytic function is advanced on the one hand by ascertaining the general form of the region of convergence (Fabry, *Comptes rendus*, 1902; Faber, *Mathematische Annalen*, 61) and on the other by developing the analytic functions of x and y in series of the form $\sum f_\nu(x)\cdot y^\nu$ (Hartogs, *Annalen*, 62). Blumenthal has studied (*Annalen*, 57) those manifoldnesses which are common to the zero-manifoldnesses of several functions.

6. Professor Stäckel showed that in many cases the Cauchy-Weierstrass inequalities can be advantageously replaced by inequalities similar in form but simpler in principle and for which it is not necessary to know the course of the function in the complex plane, but only the absolute values of the terms of the series. In this investigation of the power series for a single variable it is only necessary to assume that it converges for a value of the variable other than zero, but the corresponding assumption for power series of several variables is not sufficient. Here it is necessary to require unconditional convergence. The question whether the unconditional convergence can be deduced from less assumptions has been considered by F. Hartogs (Dissertation and Habilitationsschrift).

The speaker communicated a more general proposition, which he had proved conjointly with Hartogs, according to which it is sufficient to know that the simple series obtained by rearranging the multiple series can be made to converge uniformly in a region including the origin by a single rearrangement. Whether the conditions can be still further restricted remains undecided. A detailed account will appear in the *Jahresbericht.*

8. By aid of the theory of integral equations Dr. Hilb obtains extensions of Sturm's theorem and Klein's oscillation theorem for the equations

$$(1) \qquad \frac{d^2f}{dx^2} + k\phi(x)\cdot f(x) = 0$$

and

$$(2) \qquad \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + k[\phi(x) - \phi_1(y)]\, f(x, y) = 0$$

respectively, if in (1) within a segment of the x-axis $\phi(x)$ is sometimes positive, sometimes negative, and if in (2) $\phi(x)$ is now greater, now less than $\phi_1(y)$. Sturm's theorem then becomes: To every oscillation number m correspond a positive and a negative value for which a particular function exists that vanishes m times within the segment. Klein's oscillation theorem becomes: To every oscillation pair m, n correspond two values (positive and negative) for which a solution of (2) exists that can be expressed in the form $h(x)\cdot g(y)$, where $h(x)$ vanishes m times and $g(y)$ n times within the segments. By an extension of the method which Dr. Hilb has developed in an article "Über die Reihentwickelungen der Potentialtheorie," to appear in the next number of the *Annalen*, it follows that all the particular functions of (2) which vanish on the boundary of the rectangle formed by the two segments are identical with products $h(x) \cdot g(y)$. Hence we obtain the development in terms of such products.

9. Following some investigations of Borel, Professor Krause showed how continuous functions can be developed in uniformly convergent series of rational integral functions. He connected this with the known expressions for continuous functions in convergent series of rational integral functions and showed that under certain conditions uniformly convergent series can be assigned to the given functions.

10. Dr. Koebe proved the proposition that every triply connected plane region can be mapped conformally on a single-leaved region bounded by three circles which do not cut nor touch one another. He also gave a method of solving the corresponding problem for all n-ply connected single-leaved regions which are symmetric with respect to the real axis and have the property that every boundary cuts the real axis. The paper will appear in the *Jahresbericht*.

11. Professor Meyer extends Euclid's algorithm to rational integral functions $r_0(\lambda), r_1(\lambda), \cdots, r_{\kappa-1}(\lambda)$. The simultaneous vanishing of the $\kappa - 1$ last remainders is the necessary and sufficient condition for the existence of a variable common factor of the κ given polynomials, and the last remainder which does not vanish gives their greatest common divisor. Every one of these remainders can be expressed linearly in the given r's,

$$R_i = A_0^{(i)} r_0 + A_1^{(i)} r_1 + \cdots + A_{\kappa-1}^{(i)} r_{\kappa-1} (i = 1, 2, \cdots \kappa - 1). \quad \text{(I)}$$

An integral function of the coefficients of the r's which vanishes whenever two or more r's have a common root is called a partial resultant. Every partial resultant may be written

$$R = A_0 r_0 + A_1 r_1 + \cdots + A_{\kappa-1} r_{\kappa-1}. \quad \text{(II)}$$

If to the κ identities I and II we add the relation

$$0 = B_0 r_0 + B_1 r_1 + \cdots + B_{\kappa-1} r_{\kappa-1}, \quad \text{(III)}$$

and eliminate the r's, it follows that every partial resultant of the given polynomials, multiplied by a suitable A, can be expressed linearly in terms of the last remainders $R_1, R_2, \cdots, R_{\kappa-1}$.

$$AR = A_1R_1 + A_2R_2 + \cdots + A_{\kappa-1}R_{\kappa-1}. \qquad \text{(IV)}$$

12. In a paper presented to the Berlin Mathematical Society, Professor Schafheitlin proved that the first root of the Bessel's function of the second kind $y_n(x)$ is greater than n and that these roots obey the law obtained by Schläfli for Bessel's functions of the first kind $J_n(x)$, namely, that each root, regarded as a function of n, increases with n. In the present paper it is proved that Schläfli's proposition is also true for all the roots of dJ_n/dx and for those of dy_n/dx which are greater than the first root of y_n. For small values of n these are all the roots of dy_n/dx. It appears that the first root of d^2y_n/dx^2, for small values of n, follows that of y_n but for $n > 7.5$, two roots of d^2y_n/dx^2 precede those of y_n. Thus the curve for $y_n(x)$, after cutting the x-axis to the right of $x = n$, is convex towards it, for small values of n. But for larger values of n it has two turning points before cutting the x-axis.

13. Professor Schoenfliess's report will appear in full in the *Jahresbericht*.

14. Professor Hessenberg defines the covering x of a finite part S_1 of a class S with elements of a class R as a finite covering of S with R, and denotes by (R/S) the class of all these finite coverings. If S and R are simply ordered, (R/S) can be simply ordered and its ordinal type (ρ/σ) possesses the properties of a power of $1 + \rho$ (Hausdorff, *Leipziger Berichte*, volume 26). If further R and S are well ordered, then by the process referred to (R/S) is also well ordered and (ρ/σ) is Cantor's power $(1 + \rho)^\sigma$ which possess the defining properties

$$a^1 = a, \quad a^{\beta+1} = a^\beta a, \quad \lim_\beta a^\beta = a^{\lim \beta}.$$

The author obtains a new proof of this last proposition.

15. Professor Landsberg pointed out that the theorems of Gauss and O. Bonnet on total curvature of surfaces can be, to a certain extent, carried over to the general problem of calculus of variations. For this purpose he classifies the problems which lead to the integral

$$s = \int \phi(x, y, x', y')dt$$

according to the degree m to which x' and y' enter into the equation of condition

$$T = \phi_{x'} \cdot \delta x + \phi_{y'} \cdot \delta y = 0$$

for the transformed position. If $m = 1$ and if the relation for the new position is an involution,

$$\phi = \sqrt{Ex'^2 + 2Fx'y' + Gy'^2}.$$

Then, depending on the sign of the discriminant, we obtain Gauss's form for the element of length or one in which the minimal curves are real. In both cases the concepts of geodetic curvature of a curve at a point, curvature of the field at a point, total curvature of a region can be so stated that an extension of the above two-dimensional domain is unnecessary. It follows that the change in the length of an arc on the surface due to a uniform transverse displacement of its points is equal to the mean value of the surface curvature along this curve; if the curve is contracted to an element of arc, the change is equal to the surface curvature itself. The Gauss-Bonnet theorem on curvatura integra of a portion of a surface has also its exact analogon in the case where the minimal curves are real.

17. If from every point P of a surface of the second order the normals p to the polar plane π are drawn, a one-to-one relation is established between the axis complex p and both the space P and the space π. From these relations Professor Schmid obtains the focal properties, constructions for the normals and for the principal centers of curvature, by projecting P and p from the center O, and the other vertices of the principal polar tetraedron on the corresponding opposite faces. The substance of the paper may be found in the *Sitzungsberichte* of the Vienna Academy for 1906 (volume 115, part II *a*).

18. Professor Wiener exhibited two models for space curves of the third order, regarding such a curve 1) as the common section of two cones with a common generatrix; 2) as enveloped by two planes which touch two conics with a common tangent. In both cases we meet with the osculating tetraedron determined by two points of the curve with their tangents and osculating planes. In 1) the vertices of the cones are the two

points of the curve; in 2) the planes of the conics are the osculating planes. Each model shows the curve by aid of its tangents in strings and the osculating tetraedron by wires. They complete the set of which twelve have been exhibited and will appear with these in the Teubner series.

19. Non-analytic curves can be classified according to their order and the class and number of singular points investigated by purely geometric methods. Professor Juel treated in this way curves of the fourth order obtained as intersections of two (non-analytic) cones of the second order and a curve of the nth degree lying on a hyperboloid and cutting every generating line of one set $n - 1$ times. For both classes the same characteristic numbers were obtained as in the algebraic cases. The question was also considered whether such curves have any real existence as non-analytic.

20. In the motion of a similarly deformable plane system the usual construction for the pole — depending on the tangents to the paths at three arbitrary points of the system * — fails when these tangents intersect on the circle through the three points. Professor Müller considered this case in the light of his results in a paper "Ueber die Krümmungsmittelpunkte der Bahnkurven in ebenen ähnlichveränderlichen Systemen" (*Zeitschrift für Mathematik und Physik*, volume 36, page 129) and showed how to determine the pole and that the system admits two initial motions from the position in question.

21. Professor Runge adapted Picard's method of successive approximations to the graphic solution of differential equations of the first order. The curve is first replaced by an approximating broken line so drawn that its integral wherever it cuts the curve in the direction of the x-axis agrees with the curve integral. The curve is here replaced by arcs of parabolas with axes parallel to the axes of x or y. In this way it is possible to determine the positions of the vertical parts of the broken line so that the areas above and below the curve shall be equal. On the surface $z = f(x, y) = dy/dx$, every curve $f(x, y) =$ const. determines a direction. A rough approximation to the required curve being drawn, a piece of it AB is taken small enough so that the above directions for AB do not include too large

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* Burmester, Kinematik I, p. 867.

angles. This approximation is then made closer by using the values of f at points where AB cuts the curve $f(x, y) = \text{const.}$ If the second approximation is not close enough, the process is repeated.

23. Herr Wagenmann correlates successive steps in the theory of evolution with series $-\infty, \cdots -2, -1, 0, 1, 2, \cdots, \infty$ along three coördinate axes developing successively the ideas of motion, mass, the nebular hypothesis and evolution of living organisms and of civilization. He finds that his method leads to a monistic philosophy — in fact to a pan-monism.

A. B. FRIZELL.

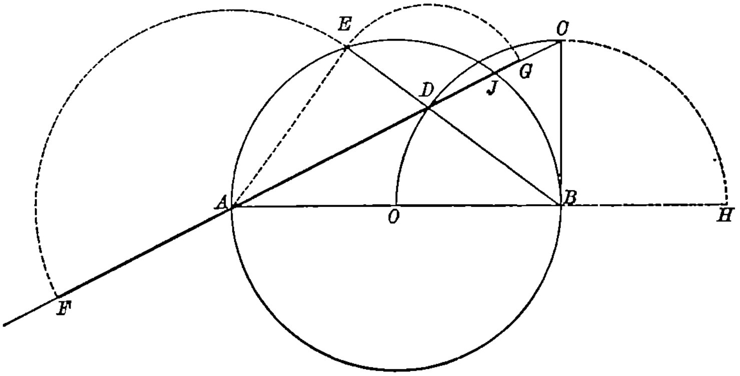
GÖTTINGEN,
November, 1906.

A NEW APPROXIMATE CONSTRUCTION FOR π.

BY MR. GEORGE PEIRCE.

GIVEN a circle with radius r and center at O; to find an approximate construction for πr.

Draw the diameter AOB and the tangent BC at right angles to it. Describe the arc ODC with radius r and center at B.



Draw the line AC cutting the arcs ODC and AB at D and J; also draw the line BDE through B and D cutting the given circle at E. Then $AD + 3DE = \pi r$ approximately.

Proof:

$$AC = \sqrt{(AB^2 + BC^2)} = r\sqrt{5}$$

$$AD = \frac{AO \cdot AH}{AC} = \frac{r \cdot 3r}{r\sqrt{5}} = \tfrac{3}{5}\sqrt{5}r,$$

$$JC = \frac{BC^2}{AC} = \frac{r^2}{r\sqrt{5}} = \tfrac{1}{5}\sqrt{5}r$$

$$DJ = AC - AD - JC = \tfrac{1}{5}\sqrt{5}r,$$

$$DE = \frac{AD \cdot DJ}{BD} = \frac{\tfrac{3}{5}\sqrt{5}r \cdot \tfrac{1}{5}\sqrt{5}r}{r} = \tfrac{3}{5}r,$$

$$AD + 3DE = \tfrac{3}{5}\sqrt{5}r + 3(\tfrac{3}{5}r) = 3.141641r.$$

By making use of the fact that in the triangle ABE

$$AE = \sqrt{(AB^2 - BE^2)} = \sqrt{(2r)^2 - (\tfrac{8}{5}r)^2} = \tfrac{6}{5}r = 2DE,$$

we can obtain a single line of the same length as $AD + 3DE$. We can therefore draw the arc EG with radius DE and center at D and the arc EF with radius AE and center at A. Then $AD + 3DE = AD + AE + DE = AD + FA + DJ = FG$.

There are many other approximate constructions for πr. A summary of those that have been worked out according to the method of geometrography is given below. A, B, C and D are to be found in the BULLETIN for January, 1902, page 137; E is in Cantor's Geschichte der Mathematik, volume 3, page 23; F is the construction given above.

			WITHOUT SQUARE.				WITH SQUARE.			
	Author.	Δ	*S.*	*E.*	Lines.	Circles.	*S.*	*E.*	Lines.	Circle
A	G. Peirce	+ .0012	22	14	4	4	17	11	4	2
B	Kühn	+ .0047	14	9	2	3	14	9	2	3
C	Lemoine	+ .0030	21	13	2	6	20	13	2	5
D	Pleskot	— .00016	24	16	3	5	24	16	3	5
E	Kochansky	— .000060	33	20	6	7	23	13	6	4
F	G. Peirce	+ .000048	24	15	4	5	19	12	4	3

Δ is the difference between the mechanically exact construction and π. S stands for simplicity and E for exactitude. For the technical meanings of these two words see the article in the BULLETIN for January, 1902. The lower these numbers are, the better the construction.

NOTE ON CONJUGATE POTENTIALS.

BY PROFESSOR O. D. KELLOGG.

(Read before the American Mathematical Society, September 3, 1906.)

IF $u(r, \vartheta)$ is a potential function of the unit circle and $v(r, \vartheta)$ its conjugate, and if $f(\vartheta)$ and $g(\vartheta)$ are the values approached by these functions as $r \doteq 1$, the following relations given by Hilbert * hold:

$$f(\vartheta) = \frac{1}{2\pi}\int_0^{2\pi} g(\phi) \cot \tfrac{1}{2}(\vartheta - \phi)d\phi + \frac{1}{2\pi}\int_0^{2\pi} f(\phi)d\phi,$$

$$g(\vartheta) = -\frac{1}{2\pi}\int_0^{2\pi} f(\phi) \cot \tfrac{1}{2}(\vartheta - \phi)d\phi + \frac{1}{2\pi}\int_0^{2\pi} g(\phi)d\phi,$$

where by the integration symbols in the first terms of the right hand sides the Cauchy principal value is meant. They give, to within an additive constant, the boundary values of a potential in terms of its conjugate.

They have been established for the case that $f(\vartheta)$ and $g(\vartheta)$ are integrable throughout and are continuous at all but a finite number of points and possess derivatives subject to the same conditions.† Because of the interest attaching to them from their connection with potential theory, and as examples of integral equations of the first kind, ‡ it seems worth while to point out that they hold if only $f(\vartheta)$ and $g(\vartheta)$ are integrable from 0 to 2π and are such that

$$\int \frac{f(\vartheta_0 + \delta) - f(\vartheta_0)}{\delta} d\delta \quad \text{and} \quad \int \frac{g(\vartheta_0 + \delta) - g(\vartheta_0)}{\delta} d\delta$$

are convergent when extended over an interval including $\delta = 0$ in its interior and this for all but a finite number of values of

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* Vorlesungen über Potentialtheorie, Gottingen, winter semester, 1901–02.

† See *Math. Annalen*, vol. 58, p. 442; also my dissertation for the doctorate: "Zur Theorie der Integralgleichungen und des Dirichlet'schen Prinzips," Göttingen, 1902, p. 17.

‡ Hilbert: "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen," *Gött. Nachrichten*, 1904, p. 49. Encykl. d. Math. Wiss. II, A, 11, pp. 803, 816.

ϑ_0 in the interval from 0 to 2π.* To justify this assertion it will only be necessary to point out how a minor change in the article in the *Mathematische Annalen* or in the dissertation just referred to permits the generalization in question.

A slightly different notation is there used: instead of the arguments ϑ and ϕ, $s = \vartheta/2\pi$ and $t = \phi/2\pi$ are employed. The reasoning may remain unchanged except for the matter of showing that the integral

$$I = \int_{-\epsilon}^{+\epsilon} g(t) \frac{\partial}{\partial t} \log \rho \left(\begin{smallmatrix} t \\ x,\, y \end{smallmatrix}\right) dt$$

is a continuous function of x and y in the neighborhood of $x = 1$, $y = 0$ (*Mathematische Annalen*, page 445; dissertation, page 21). It will be sufficient to show that this integral vanishes with ϵ in a sufficiently small neighborhood of (1, 0).

To do this, let us use instead of x and y the coördinates r and s, where $x = r \cos 2\pi s$, $y = r \sin 2\pi s$. Then $\rho^2\left(\begin{smallmatrix} t \\ r,\, s \end{smallmatrix}\right) = 1 + r^2 - 2r \cos 2\pi(s - t)$, and we may write the integral $I = I_1 + I_2$, where

$$I_1 = g(s) \int_{-\epsilon}^{+\epsilon} \frac{\partial}{\partial t} \log \rho\left(\begin{smallmatrix} t \\ r,\, s \end{smallmatrix}\right) dt,$$

and

$$I_2 = \int_{-\epsilon}^{+\epsilon} \left[g(t) - g(s)\right] \frac{\partial}{\partial t} \log \rho\left(\begin{smallmatrix} t \\ r,\, s \end{smallmatrix}\right) dt.$$

The first of these gives

$$I_1 = g(s) \log \frac{\rho\left(\begin{smallmatrix} \epsilon \\ r,\, s \end{smallmatrix}\right)}{\rho\left(\begin{smallmatrix} -\epsilon \\ r,\, s \end{smallmatrix}\right)},\dagger$$

and, as we are at a point where $g(s)$ is continuous, this factor is finite, and if we restrict the point (r, s) to that neighborhood of (1, 0) given by $0 < 1 - r < \epsilon^2$, $|s| < \epsilon^2$, the logarithm will be found to vanish at least as fast as a constant times ϵ. Hence I_1 vanishes with ϵ.

I_2 may be written

$$\int_{-\epsilon}^{+\epsilon} \left[\frac{g(t) - g(s)}{t - s}\right] \frac{2\pi r \sin 2\pi(s - t)}{1 + r^2 - 2r \cos 2\pi(s - t)} \cdot (t - s) dt.$$

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* These conditions are even less restrictive than those announced in my paper as read before the Society.

† This on the understanding that if $r = 1$ the Cauchy principal value is to be taken.

Now

$$\frac{2\pi r \sin 2\pi(s-t)}{1+r^2-2r\cos 2\pi(s-t)}\cdot(t-s)$$

is a continuous function of t for any fixed values of r and s in the above defined neighborhood of (1,0). This is at once evident when $r<1$, and becomes evident for $r=1$ when we give it the form

$$-\frac{\pi(s-t)}{\sin\pi(s-t)}\cos\pi(s-t).$$

Hence we may infer that I_2 vanishes with ϵ, for

$$\frac{g(s)-g(t)}{s-t},$$

being continuous for $t \neq s$, and having a convergent integral even over an interval including $t=0$ by hypothesis, the convergence of I_2 may be made apparent by integrating by parts, and, being convergent, it vanishes with its limits $+\epsilon$ and $-\epsilon$.

To recapitulate, the integral

$$\int_0^1 g(t)\frac{\partial}{\partial t}\log \rho(_{r,s}^{t})dt,$$

which we wished to show continuous in r and s in the neighborhood of (1, 0), breaks up into two parts

$$\int_\epsilon^{1-\epsilon} \text{ and } \int_{1-\epsilon}^{1+\epsilon},$$

or J and I as we may denote them. As I vanishes with ϵ, as we have just shown, we may so restrict ϵ and r and s that $I<\delta/4$. Then the variation of I in the neighborhood of (1, 0) thus determined will be less than $\delta/2$. But ϵ being fixed, J is continuous, and hence r and s may be so restricted that its variation is less than $\delta/2$. Thus the variation of the whole integral is less than δ, and it is therefore continuous at the point (1, 0), and that for any manner of approach whatever within or upon the periphery of the circle.

The formulas are thus established under the given conditions.

UNIVERSITY OF MISSOURI,
October 4, 1906.

GROUPS OF ORDER p^m CONTAINING EXACTLY $p+1$ ABELIAN SUBGROUPS OF ORDER p^{m-1}.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, October 27, 1906.)

IF an abelian group of order p^m, p being any prime, contains exactly $p+1$ subgroups of order p^{m-1} it has just two invariants and vice versa. Since these groups are so well known, we shall confine ourselves, in what follows, to non-abelian groups of order p^m. It is known that in every non-abelian group of this order the number of abelian subgroups of order p^{m-1} is 0, 1, or $p+1$. The present paper confines itself to the last of these three possible cases. Since any non-cyclic group of order p^m is generated by two of its subgroups of order p^{m-1}, and two such subgroups have p^{m-2} common operators, it follows that *the necessary and sufficient condition that a non-abelian group of order p^m contain $p+1$ abelian subgroups of order p^{m-1} is that it contain p^{m-2} invariant operators.* These invariant operators form a characteristic subgroup and the corresponding quotient group is of type (1, 1).

While we shall consider only non-abelian groups in the present paper, yet the abelian subgroups of order p^{m-1} will enter largely into our discussions. It seems therefore desirable to state here a few fundamental theorems with respect to abelian groups in the form in which they will be used. *If all the invariants of an abelian group of order p^m are equal, all its operators of the same order are conjugate under its holomorph and vice versa.* Any abelian group is the direct product of abelian subgroups in which all the invariants are equal. The number of complete sets of conjugate subgroups of order p^{m-1} under the holomorph of any abelian group of order p^m is equal to the number of its different sets of equal invariants, since this number is equal to the number of the different sets of conjugate subgroups of order p under the holomorph.* *The independent generators of any subgroup of order p^{m-1} in an abelian group of order p^m may be so chosen that they are either contained in a possible set of independent generators of the entire group, or that*

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* *Amer. Jour. of Math.*, vol. 22 (1900), p. 21.

all of them except one are contained in such a set while this one is the pth power of the other independent generator of this set.

The last theorem may be proved as follows: Since the number of distinct sets of conjugate subgroups of order p^{m-1} under the holomorph is equal to the number of the different sets of equal invariants of the group G it is only necessary to observe that the independent generators of one out of each such set of these conjugate subgroups can be chosen in the given manner. This may be done by constructing, in succession, the groups which result by using the pth power of one of the independent generators of G together with all its other independent generators, using the pth power of only one of the independent generators of the same order in G. The independent generators of the subgroups of lower order than p^{m-1} cannot always be chosen in this way nor is it always possible to select the independent generators of G in such a way that they become, when one of them is raised to the pth power, identical with the independent generators of a subgroup of order p^{m-1} which have been arbitrarily chosen. These statements will be illustrated in the following paragraph.

Let s_1, s_2 represent two independent commutative operators of orders p^α, p respectively and suppose $\alpha > 1$. It is impossible to choose the independent generator of the subgroup $\{s_1^p s_2\}$ of index p^2 under $\{s_1, s_2\}$ in such a manner that it either is contained in a possible set of independent generators of $\{s_1, s_2\}$ or that it is a power of such a generator. This illustrates the former of the two statements in question. If we take as the independent generators of a subgroup of index p under $\{s_1, s_2\}$, the two operators $s_1^p s_2$, s_2 the latter of these statements is illustrated, since it is impossible to select the independent generators of $\{s_1, s_2\}$ in such a manner as either to include $s_1^p s_2$, s_2 or to make $s_1^p s_2$ a power of such a generator. In fact neither of the operators $s_1^p s_2$, s_2 is a power of a larger operator contained in $\{s_1, s_2\}$.

Since a characteristic subgroup of an abelian group cannot involve any of its operators of highest order, it follows that the largest characteristic subgroup of any abelian group is composed of all its operators which are not of highest order. In particular, *the necessary and sufficient condition that an abelian group of order p^m contains a characteristic subgroup of order p^{m-1} is that it has only one invariant of highest order.*

§1. *General Properties.*

Let H represent one of the $p + 1$ abelian subgroups of order p^{m-1} and let K represent the subgroup of order p^{m-2} which is composed of the invariant operators under the entire group G. The general method which will be used in what follows is to assume H as known and then to consider all the possible groups of order p^m which contain this H. The three most important steps are: 1) the choice of K; 2) the possible transformations of H by the operators of G which are not contained in H; 3) the total number of groups which transform H in a given manner. The details of construction are given in the article * entitled "A method of constructing all groups of order p^m."

From the theorem mentioned above it follows that K can be chosen in as many distinct ways as there are different sets of equal invariants in H. That is, the number of sets of subgroups of order p^{m-2} in H which are such that each set includes all those which are conjugate under the holomorph of H is equal to the number of sets of equal invariants in H. This number will be denoted by k. As the invariants of the k distinct subgroups which are successively represented by K may be obtained by raising one of those of H, in succession, to the pth power, while the others remain unchanged, no two of these k subgroups are simply isomorphic. In particular, when all the invariants of H are equal, K can be chosen in only one way; so that $k = 1$ in this case.

We shall now consider the transformations of H by possible operators of G. In other words, we shall consider the number of different sets of operators of order p in I (the group of isomorphisms of H) which are such that each operator is commutative with all the operators of a particular K and that each set includes all the operators of I which are conjugate under the subgroup of I which transforms K into itself. As all these transformations of H may be obtained by making it isomorphic with one of its subgroups of order p and multiplying corresponding operators,† the problem is reduced to finding the number of sets of conjugate subgroups of order p in H. This number is either k or $k + 1$, as the invariant of H which is p times the corresponding invariant of K is unique or is equal to at least one other invariant of H. Under the holomorph of G the subgroups of order p in K may be contained in $k-1$ sets of conjugates, as will be seen later.

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* *Amer. Jour. of Math.*, vol. 24 (1902), p. 395.

† BULLETIN, vol. 7 (1901), p. 350.

It remains to consider the third of the three steps mentioned above, viz., the construction of the total number of groups which transform H in a fixed manner. There is clearly one and only one such G which involves an operator t of order p that is not contained in H. As K can be selected in k different ways, the total number of G's which involve H and also operators of order p which are not contained in H is $k(k+1)-l$, where l is the sum of the numbers of the sets of equal invariants of H which contain only one invariant $+1$ or 0, according as there is at least one invariant which is equal to p or no such invariant.

Each of the $k(k+1)-l$ groups considered in the preceding paragraph contains at least p abelian subgroups which are similar to H whenever $p > 2$. The invariants of the remaining abelian subgroup of order p^{m-1} are obtained by dividing one of those of H by p and adding to this quotient and the other invariants of H one which is equal to p. In the groups in which these invariants are equal to those of H the $p+1$ abelian subgroups of order p^{m-1} in G are similar. There are k such groups when H contains more than one invariant which is equal to p, $k-1$ when H contains only one such invariant, and there is no such group when the smallest invariant of H exceeds p. When $p=2$ it is still true that at least two of the abelian subgroups of order 2^{m-1} are similar but they are not necessarily similar to H.

According to the general theory of constructing the groups of order p^m, all of the other groups having the required property may be constructed by making H simply isomorphic with itself written in p distinct sets of letters and replacing t by itself written in the same systems, multiplied by a substitution which simply interchanges these systems and by an operator s from one of these systems. The number of different groups is equal to the number of distinct ways of choosing s. This choice can generally be made in $k+1$ ways, but in special cases there may be a smaller number of choices, as follows directly from the general theory. The main result which we desire to emphasize here is that since s is in K but is not a power of an operator of higher order contained in K, the remaining p abelian subgroups of order p^{m-1} are similar to each other except when just $p-1$ of them are similar to H, while the other involves more operators whose order is p times that of s than H does. We are thus led to the main theorem of this section, which may be stated as follows: *If a non-abelian group of order p^m involves $p + 1$ abelian subgroups of order p^{m-1}, at least p of these subgroups are similar to each other, and the entire group is conformal with an abelian group whenever $p > 2$.*

Since a non-abelian group of order p^3 necessarily contains exactly $p + 1$ subgroups of order p^2, the above theorem includes the theorem that a non-abelian group of order p^3 contains at least p similar subgroups of order p^2. It may also be stated that the preceding paragraph proves that the conformal non-abelian groups of order p^m which contain $p + 1$ abelian subgroups of order p^{m-1} are not necessarily identical, while it is well known that abelian groups are identical whenever they are conformal.

§ 2. *Groups in Which the $p + 1$ Abelian Subgroups of Order p^{m-1} are Similar to Each Other.*

Although these groups are included in the preceding section, yet it seems desirable to treat them somewhat more completely in view of the fact that they are of special interest. When all the invariants of H are equal to each other they are all equal to $p > 2$, or to 4. In the former case there is one and only one group of order p^m, $m > 2$. As all its operators are of order p, it is conformal with the abelian group of order p^m and of type $(1, 1, 1, \cdots)$. When all the invariants of H are equal to 4 there is one and only one group of order 2^{2n+1}, $n > 0$. Since the commutator of order 2 in such a group is the square of a non-invariant operator of order 4, it is the direct product of the quaternion group and the abelian group of order $2^{n'}$ and of the type $(1, 1, 1, \cdots)$. These results are expressed in the following theorem: *If a non-abelian group of order p^m involves $p + 1$ abelian subgroups of order p^{m-1} and if all the invariants of these subgroups are equal to the same number, then either the group is the direct product of the quaternion group and an abelian group of order $2^{n'}$ and of the type $(2, 2, 2, \cdots)$, or it is the direct product of the non-abelian group of order p^3 $(p > 2)$ which involves no operator of order p^2 and the abelian group of order $p^{n'}$ and of type $(1, 1, 1, \cdots)$*

If the conditions of the preceding paragraph are made somewhat more general by merely requiring that the invariants of each one of the $p + 1$ abelian subgroups be equal to each other, there are only two additional groups which satisfy these conditions, viz., the octic group and the non-abelian group of order p^3 which involves operators of order p^2. These properties might be employed to give new definitions of these groups. For instance, the former of these two groups, which is so fundamental in elementary mathematics, may be defined as follows: the octic group is the only non-abelian group of order 2^m which contains three abelian subgroups of order 2^{m-1} such that the invariants of each subgroup are equal to each other, but not all these invariants are equal to the same number.

We shall now consider the case where the $p + 1$ abelian subgroups of order p^{m-1} are similar to each other and where each of them involves more than one set of equal invariants. From the preceding section it follows that the invariants of H must satisfy at least one of the following two conditions: The smallest is either p or 4, or the ratio of some two is p. When the former of these conditions is satisfied, but not the latter, the invariants of K may be obtained from those of H by dropping one of the smallest ones, if these are equal to p, or by replacing a 4 by a 2. In the former case the number of possible G's, when $p > 2$, is either k or $k - 1$, according as H contains more than one or only one invariant which is equal to p. When $p = 2$ there are always $k - 1$ such G's. Finally, when the smallest invariants of H are equal to 4 there is just one such group involving a given H. This exhausts the possible cases where no two invariants of H have p for their ratio.

When H involves invariants whose ratio is p and also operators of order p or 4, the groups which have just been determined will still exist. The remaining possible groups are independent of whether H involves operators of order p or 4. When $p > 2$ there are $k + \gamma$ groups for every two sets of equal invariants which are such that the ratio of the invariants of one set with respect to those of the other is p. The value of γ is -1, 0, or 1, according as each of these two sets includes only one invariant, one includes one while the other includes more than one, or each of them includes more than one.

When $p = 2$ the remarks of the preceding paragraph remain true except when H involves invariants which are equal to 2 and also invariants which are equal to 4. In this special case it is only necessary to consider the possible groups where K does not involve all the operators of order 2, or where it involves one less invariant which is equal to 4 than H does. In the former case there are $k - 1$ distinct G's. In the latter case there are

$k+2$, $k+1$, or k such G's. The number is $k+2$ when there is more than one invariant of each of values 2 and 4 in H. When there is only one invariant of one of these values and more than one of the other, the number of G's is $k+1$. Finally, there are only k such G's when H contains only one invariant of each of the values 2 and 4.

It may be added that in the study of all the possible non-abelian groups of order p^m which contain an abelian subgroup of order p^{m-1} it is especially desirable to know all of those groups which contain more than one such subgroup, as the other possible groups are distinct whenever they transform the abelian subgroup in different ways. When there is more than one abelian subgroup of order p^{m-1} in G, two such subgroups may be transformed differently by the remaining operators.

THE UNIVERSITY OF ILLINOIS,
September, 1906.

NOTE ON SYSTEMS OF IN- AND CIRCUMSCRIBED POLYGONS.

BY MISS S. F. RICHARDSON.

(Read before the American Mathematical Society, October 27, 1906.)

IN a paper read before the London Mathematical Society on March 12, 1874 (*Proceedings of the London Mathematical Society*, volume 5) Wolstenholme assumes two similar and similarly situated polygons of n sides, $ABC\cdots KLM$ and $abc\cdots klm$, and considers the conditions for an infinity of polygons which shall be inscribed in one of the similar polygons and circumscribed about the other.

He assumes that if ab meet AM in U and if am meet AB in V, then

$$AU/AM = AV/AB = k.$$

His solution finds $n-1$ values for k, that is, that there are $n-1$ points on AM, say, which may be taken as its intersection with ab, this point fully determining the polygon $abc\cdots klm$.

In particular he finds as the two solutions for the case $n=3$ that ab must divide AC in the ratio $\frac{1}{3}$ or in the ratio 1. In the first case the triangle abc becomes a point, the common point of intersection of the lines drawn through the points of trisection of each side of ABC parallel respectively to the other two sides of the triangle. In the second case abc is a triangle circumscribing ABC, the vertices of ABC being the midpoints of the sides of abc.

For the case $n = 4$ his three solutions are that the side ab must divide the side AD in one of the ratios $\frac{1}{4}\cos^2 \frac{1}{8}\pi$, $\frac{1}{4}\cos^2 \frac{2}{8}\pi$, $\frac{1}{4}\cos^2 \frac{3}{8}\pi$, of which the middle one he says applies only to a parallelogram and reduces the quadrangle $abcd$ to a point while the other two give proper solutions and for squares give the theorem: "If $ABCD$ and $abcd$ be two concentric squares with their sides parallel and the side of one equal to the sum of the side and diagonal of the other, we can find an infinite number of quadrangles $A'B'C'D'$ such that A', B', C', D' lie on the sides AB, BC, CD, DA, and their sides $A'B'$, $B'C'$, $C'D'$, $D'A'$ pass through b, c, d, a respectively; or such that A', B', C', D' lie on the sides ab, bc, cd, da and their sides $A'B'$, $B'C'$, $C'D'$, $D'A'$ pass through D, A, B, C respectively."

He adds that in this case and for the parallelogram in general, the in- and circumscribed polygon can never, as far as he has investigated, be a convex polygon.

It will be of interest to compare the properties of the triangle and quadrangle shown by this solution with those properties shown by the synthetic solution of the general problem as proposed by Poncelet of determining the "lieu du sommet libre d'un polygone variable, dont les autres sommets parcourent des droites données, tandis que ses côtés pivotent sur des points fixes," and the "cas pour lesquels le lieu des sommets libres s'abaisse au premier degré."

Taking arbitrarily one of the polygons $ABC \cdots KLM$ and $n - 2$ consecutive vertices $b, c, \cdots, k, l$ of the other polygon, the problem of finding the remaining two vertices of the second polygon is the elementary problem of finding two centers a and m from which two projective point rows AB and LM may be projected into the same point row on a given line AM. The solution is known. The locus of the vertex a is the line MM' joining M to the point on AB which corresponds to M regarded as a point of the point row LM, and the locus of m is the line AA' joining A to the point on LM which corresponds to A regarded as a point of the point row AB, a pair of the required vertices being found as the intersections respectively of MM'

and AA' with the join of any third pair of corresponding points on AB and LM.

In the case of the triangle ABC, the point rows on AB and BC projected from the given vertex b of the triangle abc being perspective, the vertices a and c must either coincide with b or lie on a line through B. That is, the triangle abc reduces to a point or is circumscribed to ABC.*

Hence the *necessary and sufficient conditions that there be an indefinite number of triangles inscribed in one of two triangles and circumscribed about the other is that the two triangles be one inscribed in the other, the poles being the vertices of the circumscribed triangle.*

Wolstenholme's solution for the triangles is a special case of this solution.

In the case of the quadrilateral $ABCD$ and the given vertices b and c of the quadrilateral $abcd$, if the point rows on AB and CD projected respectively from the vertices b and c into the point row BC are perspective to each other, then the joins of A and B to their corresponding points on CD and the joins of C and D to their corresponding points on AB must all four meet in some point P, and the given vertices b and c must lie on some line through the point (AB, CD). They are respectively the intersections of this line with CP and BP. Similarly the vertices a and d must also lie on a line through the point (AB, CD). They may be the intersections of DP and AP with any second line through the point (AB, CD).

It follows that a sufficient condition for a system of in- and circumscribed quadrilaterals with respect to two quadrilaterals is that the two quadrilaterals be in homology, and that a pair of opposite sides of one and a pair of opposite sides of the other be concurrent.

The center of homology P, say, might be made the intersection of the diagonals of $ABCD$. And then the points d and c (or a and b) might chance to be collinear with the point (CB, DA) [assuming that by construction a and b and therefore b

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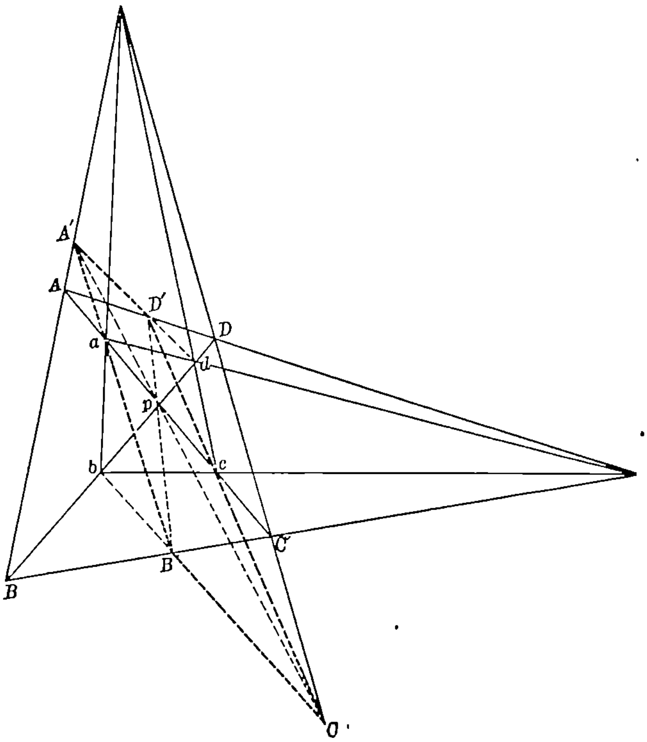
* Mr. Cayley, by his solution of the following problem proposed by Mr. W. K. Clifford: Through the angles A, B, C, of a plane triangle straight lines Aa, Bb, Cc, are drawn. A straight line AR meets Cc in R; RB meets Aa in P; Pc cuts Bb in M; MA meets Cc in r; and so on. Prove that after going twice round the triangle in this way, we always come back to the same point. (Cayley's Collected Math. Papers, volume 5, p. 589) has shown that if the vertices of the inscribed triangle be made the poles there will be closure for all points if the projection be made twice round the triangle.

and c are collinear with the point (AB, DC)]. Then the point rows AB and DC would also be perspective and the points a and b would therefore also be collinear with the point (CD, DA). This case is represented in the diagram below.

Hence, *if $ABCD$ be any quadrilateral and if through the point (BA, CD) a line be drawn meeting the diagonals AC, BD in the points a, b, respectively, and if these points be joined to the point (BC, DA) by lines meeting the diagonals BD, AC in the points d, c, respectively, then the line dc passes through the point (BA, DC) and there exists an indefinite number of quadrilaterals inscribed in $ABCD$ and circumscribed about $abcd$ if AB be projected on BC, BC on CD, CD on DA, DA on AB from the vertices c, d, a, b, respectively,* or from the vertices a, b, c, d, respectively. It is immaterial which of the two quadrilaterals be named $ABCD$.



Hence, also, *if $ABCD$ and $abcd$ be two concentric parallelograms similar and similarly placed, there exist with respect to them an indefinite number of in- and circumscribed quadrilaterals if AB be projected on BC, BC on CD, CD on DA, DA on AB from the vertices c, d, a, b, respectively, or a, b, c, d, respectively, or if ab be projected on bc, bc on cd, cd on da, da on ab from the vertices C, D, A, B, respectively, or from A, B, C, D, respectively.*

Moreover in this case *the in- and circumscribed quadrilaterals are all parallelograms;* for the point rows AB and CD as well as the point rows AD and BC are perspective, the center of projection for both pairs of point rows being the point P, the intersection of the diagonals of $ABCD$. Hence the diagonals of each of the variable quadrilaterals pass through P, since each diagonal joins a pair of corresponding points of perspective point rows whose center of projection is P. Since these diagonals are all bisected at P the variable quadrilaterals are parallelograms.

It follows from this proof that beside the system of in- and circumscribed quadrilaterals obtained by Wolstenholme with respect to the concentric squares referred to above, there is another system obtained by projecting AB on BC, BC on CD, CD on DA, DA on AB, from the vertices c, d, a, b, respectively, or from the vertices a, b, c, d, respectively, or by projecting ab on bc, bc on cd, cd on da, da on ab, from the vertices C,D,A,B, respectively, or from the vertices A, B, C, D, respectively, and that in this case the variable quadrilaterals are parallelograms instead of reentrant quadrilaterals.

If $ABCD$ be a quadrilateral and if through the point (AB, CD) lines be drawn parallel to BC and AD respectively and through the point (BC, AD) lines parallel to BA and CD respectively, these lines form a quadrilateral in homology with $ABCD$. Hence there is an infinite system of in- and circumscribed quadrilaterals with respect to these two quadrilaterals. The vertices of either may be used as poles.

If $ABCDEF$ . . . be any polygon of an even number of sides and if a, b, c, etc., be respectively arbitrary points on the diagonals AC, BD, CE, etc., there will be closure for the points A and B if AB be projected on BC, BC on CD, CD on DE, etc., from the poles a, b, c, etc. respectively. Hence if any third point on AB be projected into itself arbitrarily, the intersections of the projecting lines with AC, BD, CE, etc., will be the vertices of a polygon with respect to which and $ABCDEF$ . . . there is an indefinite number of in- and circumscribed polygons.

If $ABCDEF$ . . . be any *regular* polygon of an even number of sides and if AC intersect FB and BD in a, b respectively, CE intersect BD and DF in c, d respectively, etc., then by the properties of the regular polygon there will be closure for the midpoint of AB (and so for all points of AB) if AB be projected on BC, BC on CD, CD on DE, etc., from the poles a, b, c, etc., respectively. Also there will be closure for the midpoint of AB and so for all points of AB if AB be projected on BC, BC on CD, etc., from the midpoints of AC, BD, CE, etc., respectively. If $ABC \cdots$ have an *odd* number of sides and the projections be made from poles corresponding to those indicated in these two theorems, there will be closure for all points if the projection be made twice round the polygon.

If $ABCDE$ be any pentagon and if the points (AC, BD), (BD, CE), (DA, EC), (BE, DA), (AC, EB) be named e, a, b, c, d respectively, then if $ABCDE$ and $abcde$ be regarded as two simple pentagons there will be a poristic system of pentagons if ab be projected on bc, bc on cd, cd on de, de on ea, ea on ab from the vertices B, C, D, E, A respectively. This may be proved by testing for closure when the vertices of $ABCDE$ or of $abcde$ are the points projected.

[Professor Morley has pointed out to me that the simple pentagon $ACEBD$ and any one of the variable pentagons constitute the ten-point, ten-line configuration of Desargues's two perspective triangles.]

VASSAR COLLEGE,
October, 1906.

HERMITE'S WORKS.

Oeuvres de Charles Hermite, publiées sous les auspices de l'Académie des Sciences par EMILE PICARD. Vol. I. Paris, Gauthier-Villars, 1905. 8vo. xl + 498 pp.

I.

ON January 14, 1901, Charles Hermite passed away. A contemporary and zealous disciple of Gauss, Jacobi, and Dirichlet, the friend and generous rival of Cayley, Sylvester, and Brioschi, his death snapped one of the few remaining links connecting the present with the heroic days of the last century.

Hermite was born December 24, 1822, at Dieuze, Lorraine. His parents being in comfortable circumstances, he was early sent to continue his studies at Paris. Here at the lycée Louis le Grand, he had the good fortune to come under the charge of Professor Richard who, as may be remembered, had taught Galois in the same class of mathématiques spéciales some twelve years before.* Hermite, like Galois, early manifested an extraordinary talent for mathematics. Neglecting the regular courses of study, he read with greatest ardor the masterpieces of Euler, Lagrange, Gauss, and Jacobi; and such was the precocity of his genius that while still at Louis le Grand he published two papers and had others well under way.

In 1842, Hermite entered the Ecole Polytechnique as sixty-eighth in his class. His love for the higher mathematics had left him little leisure to prepare for examinations; hence his poor standing. His stay at this famous school was, however, destined to be short. From birth Hermite had suffered from an infirmity of the right leg and had to use a cane. On this account it was now decided by the authorities that he should not be eligible to any of the government positions which are given to the graduates of the Ecole. Hermite, therefore, left at the end of the first year. Although brief, his attendance at the Ecole Polytechnique was not without effect on his further career. For, while here, he was encouraged by Liouville to compose the first of those remarkable letters to Jacobi (January, 1843) in which the genius of Hermite shone with such extraordinary luster.

A rapid and brilliant academic career would have been the natural reward for such exceptional talents; but honors and high position were apparently to be obtained then in only one way, a path full of bitterness and humiliation to the spirit of Hermite, who had all examinations *en horreur*. It was indeed necessary for him to descend from his lofty mathematical speculations which were opening up new fields of research and occupy himself at the age of twenty-four with the petty and irksome details of preparation for the examination leading to the degrees of bachelor of letters and of science, the *licence* and the *agrégation*.

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*J. Pierpont, "Early history of Galois's theory of equations." BULLETIN, 2d Series, vol. 4, p. 335.

The first three he passed successfully. He still had the last and most trying of all before him when he fortunately received the position of examinateur d'admission and répétiteur d'analyse at the Ecole Polytechnique in 1848. These five years 1843–1848 were the Sturm und Drang period of Hermite's otherwise placid life. His appointment at the Ecole Polytechnique removed him from his trying and painful position apart from the university world, and placed him in regular standing among his friends and contemporaries. Although his advance was still by no means rapid, others with less talents receiving preference, the doubts and uncertainties which had beset his future and disturbed his meditations now melted away, and Hermite entered on his long and peaceful career with its course clearly marked out.

We close this brief sketch with a few important dates. From 1848 to 1850 he substituted at the Collége de France in place of Libri. In 1856 he was elected to the Institut. In 1862 a place as maître de conférence was created for him at the Ecole Normale. In the following year he was appointed to the responsible position of examinateur de sortie at the Ecole Polytechnique. In 1869 he became professor at this school and in 1870 at the Sorbonne. He was thus forty-seven years of age when he at length reached a position which his extraordinary talents had long claimed as their due. The professorship at the former institution he held till 1876; at the latter he remained in active service till 1897. Hermite was therefore 75 years of age when he closed his academic career.

As a lecturer at the Sorbonne, Hermite achieved a world-wide reputation. Nearly all the present generation of French mathematicians received instruction at his hands and the affectionate veneration of his former pupils was touchingly manifested on the occasion of his jubilee in 1892. Here is the tribute of one of his most gifted pupils, E. Borel: "C'est à la Sorbonne que j'ai suivi les leçons d'Hermite; c'est là que j'ai entendu cette parole si vivante exposer avec respect à la fois et avec amour les belles vérités de l'analyse. C'était un grand prêtre de la divinité du nombre qui nous en dévoilait les mystères redoutables et sacrés. Les questions les plus arides, les calculs en apparence les plus ingrats se transfiguraient, tant il avait l'intuition de leurs secrètes beautés. Quelques-uns peut-être ont eu, autant qu'Hermite, le pouvoir de faire comprendre et admirer les mathématiques; nul n'a su les faire aimer aussi profondément que lui."*

II.

Let us now turn our attention to the volume under review. This in one respect will certainly prove the most interesting one in Hermite's works. In fact, it is in these early memoirs that we may observe his wonderful genius unfold and mature; it is here that we see the germs of those fruitful notions and principles evolve with which Hermite enriched the theory of numbers, the theory of invariants and of the elliptic and abelian functions. Our astonishment and admiration grow apace as we study the contents of the 37 memoirs which make up this volume and bring us down to the year 1858. At the beginning of his career of research, Hermite seems to have actually suffered from a wealth of ideas; they overwhelm him to such an extent that he cannot take the time to develop them in order and with leisure. Rapid sketches of the main results and principles, hasty aperçus of broad horizons and fruitful fields to be exploited later, are characteristics of many of these first papers.

At the period when Hermite as a young man of twenty began to strike out for himself, the arithmetic theory of the higher forms was just beginning to be studied. Gauss had already taken the first step in the *ternary* quadratic forms when treating the composition of binary quadratic forms in the Disquisitiones Arithmeticæ. Dirichlet and Eisenstein were ably carrying on the work, when Hermite appeared upon the scene with several new and powerful principles. Chief of these are: (1) The introduction of forms with *variable* coefficients; (2) the determination and systematic employment of the upper limits of the minimum values that certain forms can take on when the variables are restricted to integral values. By their aid Hermite attacked with marked success two fundamental problems in the theory of forms, viz: the proof that the number of classes of forms of a given kind is finite, and the determination of all the linear substitutions which leave a given form unchanged. The particular forms that Hermite here studied are the quadratic forms in n variables, bilinear conjugate imaginary forms, and forms which are the product of n linear factors. While studying these profound questions, many are the applications he makes en route to problems of the most diverse nature. Such are, for example, the following: decomposition of primes into factors formed of the roots of unity; the generalization of continued fractions, leading to a criterion for cubic irrationalities, and also affording a simultaneous approximation to n given numbers; the demonstration that an analytic function of n variables cannot have more than $2n$ periods; the determination of the maximum number of fundamental units in cyclotomic bodies, the finiteness of the number of independent algebraic irrationalities belonging to the same discriminant, and the decomposition of a number as the sum of four squares. Perhaps the most notable of all is his application of the theory of quadratic forms to the theorems of Sturm and Cauchy relative to the number of roots of an algebraic equation in a given region.

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* For further details we refer to the following:

G. Darboux, "La vie et l'oeuvre de Charles Hermite," *La Revue du Mois*, Paris, 1906; vol. 1, p. 37.

E. Picard, preface to the volume under review.

M. Noether, "Charles Hermite." *Math. Annalen*, vol. 55, p. 337.

It was in the theory of numbers that Hermite first won lasting fame, and the papers on this subject not only form the larger part of the present volume, but they also have a very real importance in the problems of the present. Have we not seen what wonderful developments they were capable of in the hands of Minkowski, and are not the bilinear forms in conjugate imaginaries playing today a prominent role in automorphic functions and linear groups?

However, it was not as a disciple of Gauss and Dirichlet that Hermite made his debut in the mathematical world, but as a zealous student of the writings of Jacobi. Sixteen years before (1827) Jacobi, then an obscure privat docent at Königsberg, penned those lines, now historic, to the venerable Legendre: "Un jeune géomètre ose vous presenter quelques découvertes faites dans la théorie des fonctions elliptiques, auxquelles il a été conduit par l'étude assidue de vos beaux écrits. . . ."

No doubt with these circumstances in mind, Liouville advised Hermite to communicate some of his results to Jacobi, who now was at the zenith of his fame at the University of Königsberg. The reception and generous recognition which this letter received at the hands of Jacobi encouraged Hermite to give a further account of his researches in a second letter. Jacobi thought so highly of them that they were published in his Opuscula Mathematica as well as in *Crelle's Journal* (volume 32, 1846). The first letter was indeed a tour de force for a young man scarce twenty years of age. In 1832 Jacobi had at last succeeded in discovering the inversion of the ultraelliptic integrals and two years later in his paper "De functionibus duarum variabilium," etc., he had given a few meager indications relative to their division and transformation. This was all that was then known. The whole subject was involved in obscurity and but few mathematicians of that day had any knowledge of it whatever. Jacobi might well be surprised therefore on receiving Hermite's letter containing a complete solution of the problem of division. His second letter treats the question of the transformation of these functions, but with less success. The letter is, however, noteworthy as containing the germs of the modern theory of theta functions of one variable, of order *n* and rational characteristics, and their application to the transformation theory of the elliptic functions.

Jacobi's reply to this letter terminates with the following noble and generous passage: "Ne soyez pas fâché, monsieur, si quelques-unes de vos découvertes se sont rencontrées avec mes anciennes recherches. Comme vous dûtes commencer par où je finis, il y a nécessairement une petite sphère de contact. Dans la suite si vous m'honorez de vos communications, je n'aurai qu'à apprendre."

The problem of transformation of the ultraelliptic functions left unfinished, as just remarked, required for its solution the introduction of new elements, the theta functions in two variables and the arithmetic properties of the abelian group defined by the linear transformation of the four periods. The former was furnished by the papers of Goepel and Rosenhain (1847–1851), the latter by Hermite's own long researches in the theory of numbers. Thus ten years later, in 1855, Hermite was able to give the complete solution of the problem in a memoir which he regarded in later years with just pride. Another important application of the theory of numbers is Hermite's determination of the twenty-fourth root of unity which enters in the linear transformation of the thetas, and whose value until then was unknown. This forms the last paper in the present volume; it is moreover the forerunner of his epoch-making memoirs on the elliptic modular functions, and their application to the solution of the quintic.

To complete our picture of the rich contents of this first volume we have yet to mention Hermite's early contributions to the theory of invariants. This theory which Cayley and Sylvester had just begun to develop attracted him at first by its numerous and important relations with the higher arithmetic. Indeed, in both theories the linear transformations are of prime importance and the first invariants discovered are the determinants of quadratic forms. The present volume contains three great memoirs on binary forms, from the years 1854, 1856; their most interesting and original feature is the application Hermite makes of the principles he had already elaborated in the arithmetic theory of forms. In the profusion of results given in these memoirs we note the law of reciprocity, the existence of quadratic covariants for forms of all degrees except the biquadratic, the introduction of canonical forms, the complete system of forms for the quintic, the discovery of skew invariants, criteria for the reality of the roots of the quintic in terms of its invariants, the sextic resolvent of the quintic and the reduction of elliptic differentials of the first species to the canonical form now known as Weierstrass's.

In these investigations we see Hermite attach a particular importance to the forms of fifth degree. In fact at this time he was carrying on several lines of investigations which were to culminate in his memorable solution of the quintic. Two we have already mentioned; a third we have deferred until the present moment. Like so many of the foremost of his predecessors Hermite had been very early attracted to study the solution of the general equation of fifth degree. While still a pupil at Louis le Grand and ignorant of the work of Abel and Galois, he had published a demonstration of the impossibility of its solution by radicals, which now forms the second paper in the present volume. That Hermite still had this problem in mind we know from a letter of Borchardt, 1847. Thus when Puiseux published his memoir, 1850, on the permutation of the roots of an algebraic equation around the branch points, Hermite immediately made use of these results to develop the notion of the monodromic group of the equation and saw its application to the equations of division and transformation of the elliptic functions, 1851.

A few miscellaneous remarks in closing. As noticed above, the volumes are to appear in royal octavo and not in the customary quarto form. This first volume is furnished with a portrait of Hermite, at about the age of twenty-five; and has a preface by Picard on Hermite's scientific work. The editor gracefully acknowledges the assistance rendered by the late Professor Stouff in checking and correcting by laborious computations the results of some of the memoirs. Many of the papers bear no date and none state the pages of the volume in which they originally appeared. A moment's reflection will convince the editor how important either of these data may be. Hermite sometimes refers in a general way to some of his earlier results and it would greatly help the reader if precise references were given to the pages of the present volume. Let us illustrate. In a paper on the equation for secular inequalities Hermite (page 481), refers to certain forms he introduced "comme on pourra le voir dans un de mes Mémoires publiés dans le *Journal de Crelle*, t. 47." The reader unfamiliar with Hermite's work will naturally turn to the table of contents to find the place of this memoir in the present volume. Here he will make the unpleasant discovery that the titles of the memoirs are unaccompanied by the name and volume of the journals in which they appeared. The reader will therefore have to turn to one title page after another, until he comes to page 193 where he meets a memoir from this volume of *Crelle.* Here he finds nothing about these forms. He looks further and discovers that there is another memoir also in *Crelle* 47. This is a long paper and it takes some time to ascertain that the forms in question are not there. He may now learn that there is still a third memoir in this same volume of *Crelle,* and here on page 237 he may at last find what he so long searched for.

Again, when Hermite refers to the results of others, it would often help the reader if exact reference were given to their collected works. For example, Hermite refers, page 484, to a paper of Jacobi's in *Crelle,* volume 34. This is a misprint, as the paper is to be found in volume 36, page 97, or in Jacobi's Werke, volume 2, page 173. Or again, on page 380, Hermite refers to results "que j'ai indiquées dans le Journal de M. Thomson." As relatively few persons will know what journal is meant, why not add a footnote that it is the *Cambridge and Dublin Mathematical Journal,* and give the precise reference to its place in the present volume, viz., page 301? On pages 79 *seq.* we meet a sign which certainly will be unintelligible to some readers. It is Cauchy's symbol for the residue of an analytic function, and is, we believe, entirely obsolete now.

We have noted the following misprints: Page 165, 8 lines from top, for 46 read 40; page 315, 10 lines from top, for *foudent* read *fondent;* page 332, last line, letter missing; page 380, for *Thompson* read *Thomson*; page 449, a hyphen is missing in next to the last line, also capitalize d; page 473, for *Veierstrass* read *Weierstrass;* page 481, heading of page is wrong; page 498, 8 lines from bottom; for *snr* read *sur*.

JAMES PIERPONT.

PROJECTIVE DIFFERENTIAL GEOMETRY.

Projective Differential Geometry of Curves and Ruled Surfaces. By E. J. WILCZYNSKI. Leipzig, B. G. Teubner, 1906. viii + 295 pages.

THE present volume is the amplification and systematic development of the ideas originally presented in various papers by the author, published in the *American Journal, Transactions of the* AMERICAN MATHEMATICAL SOCIETY and *Mathematische Annalen.*

The work begins with a very brief resumé of the ideas of continuous groups, followed by a synopsis of the transformations of linear homogeneous differential equations, wherein Stäckel's theorem regarding the form of the transformations which leave such an equation invariant is generalized to apply to a simultaneous system of such equations. A fairly full discussion is given to the invariants and covariants of a single linear equation. After showing that every transformation which leaves the equation invariant is of the form

$$y = \lambda(x)\eta, \quad x = f(\xi),$$

the first transformation alone is treated at length, the functions of p_i, $p_i^{(k)}$ which remain invariant being designated as seminvariants; those of p_i, $p_i^{(k)}$, $y^{(l)}$ being called semi-covariants. By the second transformation, a function Ω of x is said to be invariant of weight m if

$$\Omega(x) = \frac{1}{(f')^m}\,\Omega(\xi).$$

An early application of these ideas is the derivation of the canonical form of the equation, in which the terms containing $y^{(n-1)}$ and $y^{(n-2)}$ are absent. The Lagrange adjoints are discussed and geometric applications made. Every solution of the adjoint furnishes an integrating factor of the equation.

Chapter III contains the first real application of the dominant idea of the book. Given a linear differential equation of order 3

$$(1) \qquad y''' + 3p_1y'' + 3p_2y' + p_3y = 0,$$

the semi-covariants z, ρ are defined by

$$z = y' + p_1y, \quad \rho = y'' + 2p_1y' + p_2y.$$

The one linear invariant is

$$\theta_3 = P_3 - \tfrac{3}{2}P_2',$$

the P_i being the coefficients in the semi-canonical form (*i. e.*, lacking y''). The quadri-derivative of θ_3 (Forsyth) is

$$\theta_{3,1} = 6\theta_3\theta_3'' - 7(\theta_3')^2.$$

Now let three independent solutions of (1) be regarded as homogeneous point coördinates in the plane. As x varies, they will define an analytic curve. The only identically self-dual curves (every tangent associated with its own point of contact) are shown to be conics. The invariants θ_3 and $\theta_{3,1}$, given as arbitrary functions of x, define a plane curve projectively. When the point $(y_1, y_2, y_3) \equiv P_y$ describes a curve, P_z, P_ρ describe other curves, semi-covariantly related to C_y. The equations of the osculating conic (having contact with C_y of the fourth order at a given point) and osculating cubic are now derived. A pencil of C_3's can be found having eight point contact. The residual basis point is called the Halphen point. In particular, it may coincide with the other eight. Such a point is called a coincidence. The cubic now has a double point and no cubic of the pencil has proper contact of the eighth order.

An interesting discussion of C_z, C_ρ follows, including a number of anharmonic properties. If all the points of C_y are coincidence points, the curve is projectively equivalent to a logarithmic spiral whose parametric angle is 30°.

If $p_1 = p_2 = 0$, C_3 superosculates C_y. If this happens for all values of x, C_y is itself a C_3. The condition that this happens is expressed in terms of invariants, and a classification of elliptic, nodal, cuspidal cubics is expressed by them. If $\theta_3 = 1$, $\theta_{3,1} = 0$, C_y is C_3.

An application of elliptic functions is then made, showing that the coincidence points on a C_3 are grouped in triangles which are inscribed in and circumscribed about the curve. Finally, the condition that C_y be an anharmonic (or *W*-) curve is also derived, and various theorems established.

Many of these results were previously obtained by Halphen and others, but the present treatment is much more systematic and complete, and may be read without a knowledge of the previous memoirs on the subject.

These ideas of transformations and invariants of a single differential equation are extended in Chapter IV to a system of linear equations. The first general paper on this subject was published by the author in the *Transactions*, volume 2, pages 1–10. This chapter embodies that paper and pages 99, 104, 106–110, 118–125 are new. Particular attention is paid to two equations, each of the second order, on account of the later application of this system. The equations are reduced to their semi-canonical form and to their canonical form, then the complete system of invariants is derived. The functionally complete system consists of four forms and those resulting from these by repetition of the jacobian process. The system of covariants is also derived. Later a precise geometric meaning is given to each of these forms, and the specializations which appear when one or more of them identically vanishes.

The part of the book that is the essentially novel work of the author commences with Chapter V, foundations of the theory of ruled surfaces. The first paper appeared in the *Transactions*, volume 2, pages 343–362.

From the simultaneous equations of the second order

$$y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z = 0,$$
$$z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z = 0$$

we can define two twisted curves C_y, C_z by letting x vary. The line (yz) will describe a ruled surface S which is projectively defined by the equations. At the torsal generators of S, the tangents to C_y, C_z intersect. By constructing the tangent planes to S at P_y, P_z we obtain the solutions of another system of linear equations, the adjoint of the given one. The two systems have the same invariants and seminvariants; they are identical if S is a quadric. When the equations are reduced to the semi-canonical form $(p_{12} = p_{21} = 0)$, C_y, C_z are asymptotic lines on S. The fundamental theorem is that when the four basic invariants are given as arbitrary functions of x, not identically zero, then S is uniquely determined, projectively considered. When three basic invariants of one system are equal to those of another, but the fourth negatively equal, the two surfaces are dualistic. If θ_4 is zero, S is identically self-dual, *i. e.*, every point is transformed into the tangent plane at that point and leaves the surface invariant.

The locus of the point through which a transversal can be drawn cutting four consecutive generators of S is the flecnode curve. It is not enveloped by the flecnodal tangents unless it be a straight line directrix. These tangents form another ruled surface, called the flecnode surface of S. As many of these properties are most easily expressed by means of line coordinates, a short resumé of line geometry of the first degree is introduced, then the ideas are applied to interpret the invariants and covariants derived before. Perhaps the most important is that the necessary and sufficient condition that a ruled surface be identically self-dual is that it belong to a non-special linear complex.

If the two branches of the flecnode curve coincide, or if the surface belongs to a special linear complex, it is not determined by means of the four fundamental invariants.

An important idea is that of the flecnode congruence, made up of the generators (of one system) of the osculating hyperboloids H of S. Its focal surface is shown to be the flecnode surface of S and various properties are established. If we put

$$\rho = 2y' + p_{11}y + p_{12}z, \quad \sigma = 2z' + p_{21}y + p_{22}z,$$

then when the line (yz) describes S, the line $(\sigma\rho)$ will describe another surface S', called the derivative of S as to x. It is contained in the flecnode congruence of S, one of its generators lying on the H of each generator of S. Various theorems are established concerning the correspondence between curves on S and on S', both for the general case and when S is contained in one or more linear complexes. If $(yz) \equiv g$, then $(\rho\sigma) \equiv g'$ is a generator of $H(g)$ of the second system, hence $H(g')$, $H(g)$ have g' in common. The residual intersection is therefore in general a space cubic, called the derivative cubic. It cuts g in two points, which with the two flecnodes on g make a harmonic range. Indeed, the idea of harmonic section, first found by Cremona to be made on the generators of certain surfaces by the asymptotic lines, and somewhat extended by other writers, is here shown to permeate the whole theory. Conditions under which the derivative cubic is composite are determined, its linear complex found and a number of properties established. The osculating linear complex of g is introduced to prove and to generalize a number of known theorems on asymptotic lines.

In case H, H' have two lines in common ($\theta_4 = 0$) the residual intersection is the derivative conic. It cannot be composite unless S has a straight line directrix. Two conics belonging to consecutive generators cannot intersect. The developable formed by the plane of the derivative conic is discussed.

We next pass to the discussion of curves upon a surface. An arbitrary curve is one branch of the flecnode curve of an infinite number of ruled surfaces, but two curves chosen at random cannot in general form the complete flecnode curve of any surface. Similarly for the complex curve and for an asymptotic line.

Chapter XIII is concerned with a space curve, defined by a single equation of order 4. Its torsal cubic, osculating cubic, conic and linear complex are treated in detail, and a careful interpretation of the special tetrahedron of reference is given. A detailed comparison with the results of Halphen is then added, with application to anharmonic curves and certain plane curves.

Throughout the book copious references are given, and but few known theorems are left uncredited. A particularly commendable feature is the collection of exercises which follows each chapter; some of them are obvious corollaries of theorems just derived, others are less directly connected, and finally a liberal number indicated by a star are unsolved problems, containing suggestions for further investigation.

The book is provided with an index and is up to the usual standard of excellence maintained by Teubner. Although full of formulas, it is singularly free from typographical errors. Of the fifteen noticed by the reviewer, only two might cause confusion. On page 55, line 22, $M_{n/2}$ should be $M_{n-2/2}$, and on page 67, line 19, Ω_2 should be Ω_1.

VIRGIL SNYDER.

CORNELL UNIVERSITY,
October, 1906.

SHORTER NOTICES.

Handbuch der Theorie der Cylinderfunktionen. Von NIELS NIELSEN, Privatdozent an der Universität Kopenhagen, Inspektor des mathematischen Unterrichts an den Gymnasien Dänemarks. Leipzig, Teubner, 1904. xii + 408 pp.

DR. NIELSEN'S treatise contains twenty-seven chapters, of which all but four are largely devoted to an exposition of his own researches. While many of the results are not new, he has given more than a score of new equations and over half as many generalizations of known theorems, also numerous new proofs. Some of these proofs are for theorems stated but not proved by Lommel, Hurwitz, Jacobi, H. F. Weber and others. In several cases his proofs are intended to supersede less rigorous proofs by Sonin, Mehler and Ermakoff. Thus the proof by Bourget, that $J^n(x) = 0$ and $J^{n+p}(x) = 0$, if n and p are integers, have no common root, is declared to be valid only for the case of multiple roots, a proof also criticised by Rayleigh.

A cylindrical harmonic is defined as a solution of two functional equations which are shown to lead to Bessel's equation. The general solution of the first fundamental equation is obtained, and from a new property of the second equation follows a solution in the form of a continued fraction. The influence which Kepler's equation has exerted upon the study of cylindrical harmonics is recognized, but the problem from which the functions obtained their name appears to have been overlooked. The author would advise dropping the name Bessel function, because Bessel used only integral parameters, also because Bernoulli, Euler, Laplace, Fourier, Poisson and others had previously known of them; but he has consented to call J^n the Bessel cylindrical harmonic and Y^n the Neumann cylindrical harmonic because of the fundamental work done by their respective investigators.

Dr. Nielsen has aimed to obtain generalized forms and theorems, and with this in view he has devoted considerable space in the first of the four parts of his work to Lommel's Π function and to the similar Φ function, thus laying a firm foundation for a new theory of definite integrals with cylindrical harmonics and for Schlömilch's series. In this part are given also new developments of Krampe's integral $\int_0^x \epsilon^{-x^2}dx$, and two of $J^n(x)$.

A chapter is also given to the integration of Riccati's equation, an equation of the third order and one of the fourth order by methods which may be extended to those of higher order.

Although this treatise does not go into practical applications, the author deviates from this course by giving at the end of the first part a list and brief description of tables of Bessel's functions mostly J^0 and J^1, including one of J^n, also one of Y^0 and Y^1. He might have included a table by Dr. Meissel giving the first fifty roots of $J(x) = 0$ and the corresponding maximum or minimum values of $J^0(x)$, which may be found in Gray and Mathews's Bessel Functions, together with the formulas by Professor J. McMahon in this connection.

The new theory of definite integrals mentioned above is given in the second part, and though it would have been possible to express the results as particular integrals of the general differential equation, the author has preferred not to do so, and has thus avoided introducing a large class of more general functions. Attention is called to an incorrect equation by Struve which is quoted by Gray and Mathews, page 238, example 46. The latter have made one correction, but the denominator of the integrand as printed should be raised to the nth power.

The third part is devoted to the development of analytic functions by means of cylindrical harmonics and gives a new analogy between Neumann's series of the first and second kind; also two new developments of elliptic integrals of the first kind. An interesting result is a new addition formula, $J^n(x+y)$ in an infinite series of terms each containing a product of two Bessel functions.

The development of arbitrary functions in terms of cylindrical harmonics is the topic of the fourth part. A series expansion by Lommel, proceeding according to Bessel functions of ascending orders, each differentiated the same number of times, does not appear. A new solution of Kepler's equation is given, believed to have advantages in practical application.

At the end of the book is a collection of more than sixty formulas or theorems under the headings: gamma functions, hypergeometric series, spherical harmonics, etc., also a few notes upon portions of the text. These theorems are referred to by number in order to avoid digressions in the course of the proofs. Here appears a note concerning an important misprint in Dini's discussion of Fourier's series, referring also to a letter from Dini. Earlier in the text Dini was mentioned as the only one among several writers who had given a rigorous proof of the correctness of the expansion of $f(x)$ in a series of terms each involving a Bessel function. This expansion caused Todhunter to state that many German writers credit Fourier with its authorship, though in fact he did not give it; and into this category it may be inferred that Nielsen has fallen.

Last of all comes a most important part of the book, a very complete bibliography, giving references to both theoretical and applied work in cylindrical harmonics. In addition, at the bottom of many pages are references to the original sources of nearly all formulas, in many cases proved by methods different from those in the text. A paper by Glaisher on Riccati's equation appeared in *Philosophical Transactions* in 1881, not in 1882, while to Schläfli's credit may be added an extensive article in *Annali di Matematica,* series 2, volume 6. In 1867 Lommel mentioned nine writers, while in this list appear one hundred and fifty-five.

With all the work which Dr. Nielsen has brought within the compass of a volume of moderate size, and in which he has had so great a share, there remain unexplored fields. Apart from his frank statement that we do not know the necessary and sufficient conditions under which a function is developable in a Fourier's series, there are other topics more closely related to the text, such as the remainder terms in null developments, and the single valuedness of developments in a Schlömilch's series, also many topics not fully treated in the present work.

F. H. SAFFORD.

Space and Geometry in the Light of Physiological, Psychological and Physical Inquiry. By Dr. ERNST MACH. Translated from the German by THOMAS J. MCCORMACK. Chicago, The Open Court Publishing Company, 1906. 148 pp.

To appreciate this work it is necessary to view it in its relation to two complementary movements in modern mathematical thought, namely, the logical movement and another that may be significantly called biological. The aim of the former has been to detect and to enumerate all definite ideas or terms that are indefinable and all definite propositions that are indemonstrable, to combine these primitives into all logically available sets of compatibles, and then by processes of pure inference to render explicit in systematic form the various contents implicit in the various sets. The method has been that of abstraction, postulation and deduction; the concern has been, not with external validity or applicability, but solely with consistence, with logical coherence, with conceptual and propositional harmony; and the effect has been more and more to eliminate intuition, to detach mathematics from experience, from reality, from life, from the sensuous world of things and events. The advance has followed two paths, the path of the well-known mathematical rigorists, and the path of symbolic logic under the leadership of C. S. Peirce, Schröder, and especially Peano; and these paths unexpectedly to most have been found to converge in the remarkable thesis that pure mathematics is symbolic logic and that pure logic is symbolized mathematics. A notable mark of the movement has been a strong tendency to nominalism, as witness, for example, Hilbert's Fundamental principles of geometry, in which the point, the line and the plane are nothing but names of undefined entities (replaceable by other names or entities or both) satisfying a prescribed system of postulates.

Meanwhile another movement has been going on and in recent years rapidly gaining in interest and force. It contains two principal components. These are easily confounded because they agree in seeking to reattach mathematics to experience, in demanding a reunion and a more intimate union than ever before of mathematical science and reality. Nevertheless the two components are entirely distinct. They differ radically in respect to the kind of union they severally contemplate. The one aims to establish or to reestablish a union supposed never to have existed or to have been broken up. The other aims to discover a union supposed to have existed always as in the nature of things. The former is partly due to a reaction against the logical movement, the tendency to nominalism, and partly to the increasing demand of an increasing number of sciences such as engineering, chemistry, anthropology, economics, psychology and statistics, that mathematics shall cease to dwell apart, interested exclusively in its own evolution as a pure science, and adapt itself, its teaching and investigation, to their needs. The latter component is not of the nature of such reaction and is not due to such a demand. Its aim. is to unite mathematics and experience, not in the sense of rendering the science applicable in other fields of investigation, but in the sense of showing that mathematical concepts, however tenuous or pure or remote or recondite, have been literally evolved continuously in accordance with the needs of the animal organism out of the elements (feelings) of physiological experience. This is why the second movement may properly be described as biological. The extent to which the movement may be destined to succeed is a question that only time can answer. The enterprise is undoubtedly legitimate and is one that men as rational beings were bound sooner or later to undertake. The significance of it, its bearings on theory and on practice, on teaching in particular, and on the conception that the future mathematician may have of his science and of its relations to other modes and forms of intellectual activity, can scarcely fail to be profound.

The leading contributor to the biological movement in mathematical criticism, at least the most widely known contributor, is Professor Mach, whose Science of mechanics, Popular scientific lectures, and Contributions to the analysis of the sensations are well-known and are highly valued in scientific and critical circles everywhere throughout the western world. The book in hand ought to be read and pondered by every teacher of mathematics and by every educated guardian of the young. Physiological space, the space of vision, the space of touch, the space of audition, metrical space, the space of geometry, the correspondences, likenesses and differences of these, feelings of constance, of motion, of velocity, of acceleration, of locations, the interplay and biological functions of these sensations, their contributions to geometry,— such are some of the themes of the discussion, which, running from simple facts of vision through a critical characterization and comparison of the chief varieties of metageometry, is remarkable alike for the questions it propounds and for the answers it gives or suggests. The Kantian philosopher will find here ample reason to reconsider his master's doctrine of space, the famous saying of Kant that "Thoughts without contents are empty, intuitions without concepts are blind" being happily transformed by Mach into "Concepts without intuitions are blind, intuitions without concepts are lame." The indications are clear that the psychologist of the future will, contrary to the rash predictions of Dr. Hall, find a rich field for psychological research in the concepts of mathematics; and the mathematician in his turn will not fail to find evidence that the purest offspring of his thought may trace a legitimate lineage back and down to the rudiments of physical and physiological experience. The author's discourse carries waters from numerous confluent sciences, and is a living witness to the unity of knowledge.

The translation, as one knowing Mr. McCormack's previous work would expect, is well-nigh perfect. The Open Court Company is again to be congratulated on its excellent judgment and on its generosity in the service of science.

C. J. KEYSER.

NOTES.

WITH the present issue of the BULLETIN Professor H. S. WHITE retires from the editorial staff, to assume larger responsibilities as member of the Editorial Committee of the *Transactions*. The Committee of Publication takes this opportunity to express its sense of obligation to Professor White for his valuable editorial support during the past two years.

A MEETING of mathematicians interested in the formation of a Southwestern Section of the AMERICAN MATHEMATICAL SOCIETY was held at Columbia, Mo., on Saturday, December 1, about thirty-five persons being in attendance. Over twenty papers were presented. Steps were taken toward permanent organization. A full report of the meeting will appear in the BULLETIN.

THE concluding (October) number of volume 28 of the *American Journal of Mathematics* contains the following papers: "Functions of three independent variables (concluded)," by H. L. COAR; "An invariant condition for certain automorphic algebraic forms," by A. B. COBLE; "On some cases of motion of a solid in infinite liquid," by G. KOLOSOFF; "On the arrangement of the real branches of plane algebraic curves," by V. RAGSDALE.

THE January number (volume 8, number 2) of the *Annals of Mathematics* contains: "Circles orthogonal to a given sphere," by C. L. E. MOORE; "On functional determinants," by PAUL SAUREL; "Involutory transformations in the projective group and its subgroups," by E. B. WILSON; "On the convergence and differentiation of certain classes of trigonometric series," by W. C. BRENKE; "Note on the definition of an abelian group by independent postulates," by W. A. HURWITZ.

AT the annual meeting of the London mathematical society held on November 8, the following officers were elected: W. BURNSIDE, president; A. E. H. LOVE and J. H. GRACE, secretaries; J. LARMOR, treasurer; two vice-presidents and ten other members of the council. The following papers were read: By A. R. FORSYTH (presidential address), "Partial differential equations, some criticisms and some suggestions";

by A. C. DIXON, "Harmonic expansions of functions of two variables"; by G. N. WATSON, "General solution of Laplace's equation in n dimensions"; by H. HILTON, "On subgroups of a finite abelian group"; by J. E. CAMPBELL, "On Bäcklund's transformation and the partial differential equation $s = F(x, y, z)$"; by H. BATEMAN, "Inversion of a double integral."

THE annual meeting of the National academy of sciences was held at Harvard University, November 20–21.

THE seventh meeting of the Association of teachers of mathematics in the Middle States and Maryland was held at the Central high school, Philadelphia, on December 1, 1906. After an address of welcome by R. E. THOMPSON, president of the Central high school, the following papers were read: By CHARLOTTE F. MCLEAN: "Some suggestions for relieving the strain in elementary mathematics"; by HARRY ENGLISH: "The teacher — his preparation, place, and power'; by A. G. RAU: "Co-ordination in mathematics"; by JOHN MILLER: "The fundamental theorems and elementary mathematics"; by ALICE M. MCKELDEN: "The problems that arise in the teaching of elementary algebra"; by F. H. SAFFORD: "Inversion."

The following officers were elected for the coming year: president, E. S. CRAWLEY; vice-president, J. S. FRENCH; secretary-treasurer, J. T. RORER.

THE Central association of science and mathematic teachers, acting with the American society of teachers of mathematics and the natural sciences, has issued a call for a meeting of delegates of associations of like nature, to be held at Columbia University on December 27, for the purpose of discussing the question of the formation of a national federation of teachers of mathematics and science.

THE annual meeting of the New York state science teachers association was held at Teachers College, Columbia University, New York, on December 26–27. Several papers on mathematical topics were presented.

THE following books are announced in the press of B. G. Teubner, Leipzig, and will probably appear in a few weeks: Encyklopädie der Elementar-Mathematik, volume 3, by H.

WEBER and J. WELLSTEIN; Vorlesungen über Geschichte der Mathematik, volume 4 (1758–1800), by M. CANTOR; Theorie des Integrallogarithmus und verwandter Transzendenten, by N. NIELSEN; Vorlesungen über die Elemente der Differential- und Integralrechnung, by H. BURKHARDT; Vorlesungen über Zahlentheorie, by J. SOMMER; Fragen der Elementargeometrie, volume 2, by F. ENRIQUES, translated by H. FLEISCHER; Synthetische Geometrie der Kegelschnitte, by P. SCHAFHEITLIN.

AMONG the new models which have just been added to the list of Schilling in Halle are an elliptic circle, by K. ROHN; a wooden model to illustrate the Dandelin theorem, by E. KÖTTER, and a metallic one to demonstrate the generation of an ellipse in space, by C. HILDEBRANT; three dimensional nets of four dimensional bodies, by R. GAETSCHENBERGER; and a plaster model of the locus of the center of a chord of a twisted quartic curve, by D. BÖHMLÄNDER.

THE new rules regarding the mathematical tripos of Cambridge University, as sanctioned by the senate on October 25 are as follows: (1) A student may be a candidate for part I at a date not earlier than the second term nor later than the seventh term; (2) A student having failed to obtain honors may try a second time, upon accepted recommendation; (3) The subjects must be chosen from the schedule annexed to the report; (4) The list of successful candidates will be published in three classes, each arranged alphabetically; (5) The subjects for part II must be from A, B of report, and part I; (6) The successful candidates will be announeed as wranglers, senior optimes and junior optimes, each arranged alphabetically; (7) The class will be determined by the proficiency shown in A, provided a fair showing is made in B. (Cf. BULLETIN, volume 12, page 468 and volume 13, page 131.)

OXFORD UNIVERSITY (*Hilary term*, 1907).—By Professor W. ESSON: Comparison of analytic and synthetic methods in the theory of conics, two hours; Synthetic properties of cubics, one hour.—By Professor E. B. ELLIOTT: Elements of elliptic functions, two hours; Theory of numbers (continued), one hour.—By Professor H. H. TURNER: Mathematical astronomy, two hours.—By Professor A. E. H. LOVE: Theory of potential, two hours; Calculus, two hours.—By Mr. P. J.

KIRKBY: Higher plane curves, two hours.—By Mr. A. L. DIXON: Calculus of finite differences, two hours.—By Mr. J. E. CAMPBELL; Differential equations, II, two hours.—By Mr. C. H. SAMPSON: Solid geometry, two hours.—By Mr. J. W. RUSSELL: Determinants, two hours.—By Mr. C. LEUDESDORF: Geometry of inversion, two hours.—By Mr. A. E. JOLLIFFE: Analytical geometry, two hours.—By Mr. R. F. McNEILE: Integral calculus, two hours.—By Mr. E. H. HAYES: Elementary mechanics, three hours.—By Mr. C. H. THOMPSON: Dynamics of a particle, two hours.

AMONG the medals presented at the annual meeting of the Royal society of London was a royal medal to Professor A. G. GREENHILL for his researches in the applications of elliptic functions.

AT the Stuttgart meeting of the Deutsche Mathematiker-Vereingung the sum of 933.80 marks was set apart to restore and preserve the grave of RIEMANN, who was buried at Biganzolo, Italy, in 1866.

AT the University of Maine, Mr. E. E. MOOTS has been appointed instructor in mathematics.

CATALOGUES of second-hand mathematical works: Gustav Fock, Schlossgasse 7, Leipzig, Germany, Antiquariats-Verzeichnis no. 293, 4401 titles in mathematics and physics. — W. Junk, Kurfüstendamm 201, Berlin, Bulletin nos. 1–2, 163 titles in mathematics.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Baker (W. M.). Algebraic geometry. New treatise on analytical conic sections. London, 1906. 8vo. 338 pp. Cloth.

Bourdon. Application de l'algèbre à la géométrie, comprenant la géométrie analytique à deux et à trois dimensions. 9e edition. Paris, Gauthier-Villars, 1906. 8vo. 18 + 650 pp. F. 9.00

Bromwich (T. J. I'A.). Quadratic forms and their classification by means of invariant factors. Cambridge, University Press, 1906. 8vo. 8 + 100 pp. (Cambridge Tracts in mathematics and mathematical physics, No. 3.) 3s. 6d.

Cohen (A.). Differential equations. Boston, Heath, 1906. 8vo. Half leather. $2.00

Czuber (E.). Vorlesungen über Differential- und Integralrechnung. (In 2 Bänden.) Vol. II. 2te, sorgfältig durchgesehene Auflage. Leipzig, Teubner, 1906. 8vo. 8 + 528 pp. Cloth. M. 12.00

Demuth (W.). Ueber unendliche Potenzreihen, deren Koeffizienten nach arithmetischen Reihen hoherer Ordnung fortschreiten. (Progr.) Blankenburg, 1906. 4to. 20 pp.

Durège (H.). Elemente der Theorie der Funktionen einer komplexen veranderlichen Grösse. 5te Auflage, neubearbeitet von L. Maurer. Leipzig, Teubner, 1906. 8vo. 10 + 397 pp. M. 9.00

Ehrat (J.). Ueber das Algebraischwerden der Integrale irrationaler Differentiale von der Form $\pi(x, y)dx$, in welchen $\pi(x, y)$ eine rationale Funktion ist und zwischen x und y eine allgemeine Gleichung zweiter Ordnung besteht. (Diss.) Zurich, 1906. 8vo. 50 pp. M. 2.50

Enriques (F.). Problemi della scienza. Bologna, 1906. 8vo. 593 pp. L. 10 00

Gottschalk (A.). Zur Integration der Eulerschen Differentialgleichungen. (Progr.) Gronau, 1906. 8vo. 7 pp.

Grünwald (A.). Betrachtung von Fusspunktskurven in der Ebene und im Raume. (Progr.) Prag, 1906. 8vo. 34 pp.

Hass (E.). Merkwürdige Kurven im Dreieck und ihre Beziehung zu den sogenannten "merkwurdigen Punkten." (Progr.) Duisburg, 1906. 4to. 27 pp.

Isenkrahe (C.). Ueber die zweiunddreissig Lösungsergebnisse des erweiterten Malfattischen Problems. (Progr.) Trier, 1906. 8vo. 46 pp.

Juel (C). Om ikke-analytiske kurver. Kjöbenhavn, 1906. 4to. 78 pp. Kr. 1.95

Junker (F.). Höhere Analysis. Teil I: Differentialrechnung. Leipzig, Goschen, 1906. 16mo. 204 pp. (Sammlung Göschen.) M. 0.80

Kienast (A.). Ueber die Darstellung der analytischen Funktionen durch Reihen, die nach Potenzen eines Polynoms fortschreiten und Polynome eines niederen Grades zu Koeffizienten haben. (Diss.) Zürich, 1906. 8vo.

Kistler (H.). Ueber Funktionen von mehreren komplexen Veränderlichen. (Diss., Gottingen.) Basel, 1905. 8vo. 45 pp.

Leschanowsky (H.). Gemeinverständliche erste Einführung in die hohere Mathematik und deren Anwendung. Wien, Fromme, 1906. 8vo. 8 + 85 pp. M. 2.50

Lütkemeyer (G.). Der analytische Charakter der Integrale von partiellen Differentialgleichungen zweiter Ordnung in Anwendung auf die Theorie der Flachenverbiegung. (Progr.) Gelsenkirchen, 1906. 4to. 29 pp.

Maillet (E.). Introduction à la théorie des nombres transcendants et des propriétés arithmétiques des fonctions. Paris, Gauthier-Villars, 1906. 8vo. 5 + 280 pp. Fr. 10.00

Mauderli (S.). Die Interpolation und ihre Verwendung bei der Benutzung und Herstellung mathematischer Tabellen. (Progr.) Solothurn, 1906. 8vo. 146 pp.

Maurer (L.). See Durège (H.).

Miller (G. A.). Groups generated by two operators which transform each other into the same power. 8vo. 4 pp. (*Prace matematyczno-fizyczne*, vol. 17, pp. 119–122.)

Müller (F. J.). Abbildung eines Sphäroidstreifens auf die Ebene. (Diss.) Würzburg, 1906. 8vo. 27 pp.

München (K.). Die Krümmungslinien auf der windschiefen Schraubenfläche. (Progr.) Feldkirch, 1906. 8vo. 47 pp.

Niewenglowski (G. N.). Les mathématiques de la médecine. Paris, 1906. 8vo. F. 2.50

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THE PRELIMINARY MEETING OF THE SOUTHWESTERN SECTION.

THE preliminary meeting of the proposed Southwestern Section * of the AMERICAN MATHEMATICAL SOCIETY took place at Columbia, Mo., on the 1st of December, 1906. About thirty-five persons, including the following members of the Society, were present:

Professor L. D. Ames, Mr. R. L. Börger, Professor A. S. Chessin, Brother Constantius, Professor G. R. Dean, Dr. E. L. Dodd, Dr. F. J. Dohmen, Mr. E. S. Haynes, Professor E. R. Hedrick, Professor O. D. Kellogg, Mr. W. A. Luby, Professor E. H. Moore, Professor Alexander Pell, Professor Oscar Schmiedel, Mr. F. C. Touton, Miss M. S. Walker, Dr. Paul Wernicke, Dr. W. D. A. Westfall, Mr. A. M. Wilson, Professor J. W. Withers.

The meeting was called to order by Professor Hedrick at 10:30 A. M. Professor Hedrick was elected chairman and Professor Chessin secretary of the meeting. Professor E. H. Moore was asked to preside as honorary chairman of the meeting for the afternoon session. The morning session adjourned at 1 P. M.

The meeting was called to order for the afternoon session at 2:30 P. M. The reading of papers was preceded by a short business meeting. On the motion made by Professor Chessin the members present unanimously selected St. Louis as the next meeting place of the proposed Southwestern Section.

A programme committee was elected, consisting of Professor E. R. Hedrick, chairman; Professor A. S. Chessin, secretary; Professor M. B. Porter.

At the conclusion of the meeting a motion was passed expressing the thanks of the members to all who had assisted in the formation of this section, especially to Professor E. H. Moore. The meeting adjourned at 5:45 P. M., and was followed by an informal supper at which about twenty were present.

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* The formation of the Southwestern Section was authorized at the meeting of the Council on December 28.

The following papers were read:

(1) Professor L. D. Ames: "Notes on the orientation of a secant."

(2) Mr. R. L. Börger: "On the determination of the subgroups of the special linear ternary homogeneous group in the Galois field p^2" (preliminary report).

(3) Professor A. S. Chessin: "On a particular case of motion of Sire's polytrope and Foucault's gyroscope."

(4) Professor G. R. Dean: "The equations of motion of a compressible frictionless fluid" (preliminary report).

(5) Dr. E. L. Dodd: "Incomplete double sequences as exemplified by infinitesimals."

(6) Professor S. Epsteen: "The probable error of a measurement a unit in length."

(7) Dr. O. E. Glenn: "On a class of operation groups of order $p_1^{m_1\mu_1}p_2^{m_2\mu_2}$."

(8) Professor O. D. Kellogg: "Harmonic functions on the boundary of their region of definition."

(9) Professor Oscar Schmiedel: "The polynomial coefficient generalized and extended."

(10) Professor E. H. Moore: "Homogeneous distributive functional operations of degree n."

(11) Professor H. B. Newson: "Real conformal transformations in space."

(12) Professor G. A. Miller: "Groups in which every subgroup is either abelian or dihedral."

(13) Professor M. B. Porter: "Change of variable in a multiple integral."

(14) Professor M. B. Porter: "Pringsheim's criterion for expansibility in Taylor's series."

(15) Dr. Paul Wernicke: "On the interlocking and knotted curves and surfaces."

(16) Dr. W. D. A. Westfall: "Generalization of the fundamental theorem of Fourier constants."

(17) Professor E. H. Moore: "Notes on Fourier's constants."

(18) Professor L. E. Dickson: "Invariants of binary forms under modular transformations."

(19) Professor Alexander Pell: "Some remarks on curves of constant torsion."

(20) Professor M. B. Porter: "Polynomial convergents of a power series."

In the absence of the authors Dr. Glenn's paper was read by Professor Kellogg, Professor Dickson's by Professor Hedrick, Professor Porter's third paper by Dr. Westfall, and Professor Miller's paper and Professor Porter's first two papers were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In certain problems in the theory of potential it is desirable to be able to define uniquely the positive sense of a secant to a smooth curve, and also to define the angle it makes with a given line as a single valued continuous function of the points of intersection. Professor Ames showed that this is conveniently done by means of the definition of an angle from its sine and cosine taken together, as used in a previous paper (*American Journal of Mathematics*, October, 1905, page 349).

2. In Mr. Börger's paper the subgroups of order a power of p and those of order a multiple of p down to the order p^3M (M being prime to p) are determined for the special linear ternary homogeneous groups in a Galois field p^2.

3. When the mass of the rings in the apparatus is neglected the motion of Sire's polytrope or Foucault's gyroscope is expressible by means of elliptic or circular functions. Professor Chessin showed that the solution could be also effected in terms of elliptic functions in the case when $A_1 = A + C_1$; A_1, A_1, C_1 and A, A, C indicating respectively the principal moments of inertia of the internal ring and of the torus. Such a relation may be obtained by a proper construction of the gyroscope.

4. For the case of steady motion in two dimensions under no forces we have the equations

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \quad u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

and the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{1}{\rho}\left(u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y}\right).$$

Eliminating p and ρ by means of the isothermal relation $p = k\rho$, Professor Dean obtains an equation of the second order of Monge's form which is easily integrated, and the arbitrary functions may be determined to fit given boundary conditions. The adiabatic relation also gives a Monge equation.

If the acceleration due to gravity is introduced, another term is introduced into the equation, rendering the subsidiary system of pfaffians inexact, *i. e.*, not capable of being represented by a single integral equation.

In the case of motion in three dimensions, both the isothermal and adiabatic relations give equations in three independent variables analogous to Monge's. In the case of no forces the system of pfaffians is exact, but when forces are introduced the system does not satisfy the conditions of integrability.

5. Dr. Dodd notices that the totality of the infinitesimals α_{in}, when $n \doteq \infty$, forms an incomplete double sequence, which lacks the terms on one side of the main diagonal. A common error in the statement of a theorem regarding the substitution of infinitesimals in an infinite sum may be made clear by two sequences whose terms are: $\alpha_{in} = 1/n$ and $\beta_{in} = 1/(n+1-i)$ respectively. In this case,

$$\lim_{n=\infty} \beta_{in}/\alpha_{in} = 1\,;\ \lim_{n=\infty} \sum_{i=1}^{i=n} \alpha_{in} = 1. \quad \text{But} \lim_{n=\infty} \sum_{i=1}^{i=n} \beta_{in} = \infty\,.$$

6. Basing his argument on an assumption which differs from the one used in Merriman's Least Squares, Professor Epsteen obtains a different value for the probable error of a measurement a unit in length, viz., $0.6745\sqrt{\sum pd^2(2n-1)^{-1}}$ where $p = 2(l + l')^{-1}$, and $d = l - l'$, l and l' being duplicate measurements.

7. The first part of Dr. Glenn's paper was published in full in the BULLETIN, May, 1906. In the second part he extends the methods there employed to include the determination of the defining relations of a class of groups of the given order, whose subgroup of order $p_1^{m_1\mu_1}$ is the abelian type $[\mu_1, \mu_1, \cdots, \mu_1]$, $(\mu_1 > 1)$. A general method of determining the number of distinct types in these relations is developed.

8. In this paper Professor Kellogg extends the investigations announced at the September meeting of the Society to

regions other than the circle. An apparently new view point is attained in seeking conditions for the existence of the derivatives on the boundary which bear as nearly as possible equally upon boundary curve and boundary values. Restricting the problem to normal derivatives the following result is obtained: Let $x = x(s)$, $y = y(s)$ be the parameter equations of the curve, s being the length of arc between a fixed and a varying point, and let $f(s)$ be the boundary values to be approached by the potential function. Then if three positive constants A, a and δ can be determined, such that for every s $|\Delta x'(s)| < A\,|\Delta s|^a$ and $|\Delta y'(s)| < A\,|\Delta s|^a$ as soon as $|\Delta s| < \delta$; and if the integral

$$\int \left| \frac{f(s+t) - f(s)}{t} \right| dt$$

is convergent for every value of s when extended over an interval containing $t = 0$ in its interior, then the derivative of the potential function formed in the direction of a normal to the boundary curve approaches a finite limit as the boundary is approached along the normal, and these limits form a continuous function on the boundary. Incidentally a generalization of Fredholm's proof of the Dirichlet principle is obtained, the boundary curve being subject only to the above restriction and to being a single closed curve without double points; and some new light is thrown upon the potentials of double distributions on a curve.

9. The binomial coefficient represented in Grassmann's notation by $n^{\cdot a}$ is generalized by Professor Schmiedel to embrace negative values of a and extended, the symbol $n^{\cdot a \cdot b \cdot c \cdots r}$ being introduced. The fundamental relations of the extended coefficients are derived and numerous applications are shown, notably the forming of the nth derivative of a function of several variables.

10. In generalization of current concepts of linear distributive functional operations Professor Moore proposes the following definition:

On the supposition that (a) the notations u relate to elements of a field F having no modulus; (b) the notations ϕ relate to elements of a class $\mathfrak{M}$ linear with respect to F, *i. e.*, such that the notations

$$u\phi = \phi u, \quad u_1\phi_1 + u_2\phi_2 = u_2\phi_2 + u_1\phi_1, \quad 1\phi = \phi, \quad 0\phi = 0,$$

$$(u_1+u_2)\phi = u_1\phi + u_2\phi, \ u(\phi_1+\phi_2) = u\phi_1 + u\phi_2, \ u_1(u_2\phi) = (u_1u_2)\phi$$

specify in each case definite elements of $\mathfrak{M}$; and (c) the notations ϕ^* relate to elements of a class $\mathfrak{M}^*$ linear with respect to F:

An operation A assigning to every ϕ a definite corresponding $\phi^* = A(\phi)$ is called a homogeneous distributive operation of degree n effective in a single valued way from $\mathfrak{M}$ to $\mathfrak{M}^*$ with respect to F, in case the $n + 2$ elements

$$\psi_h^* = A(\psi_h) \qquad (h = 0, 1, \cdots, n + 1)$$

of $\mathfrak{M}^*$ corresponding to $n + 2$ elements

$$\psi_h = u_{h1}\phi_1 + u_{h2}\phi_2 \quad (h = 0, 1, \cdots, n + 2)$$

in a binary linear system (ϕ_1, ϕ_2) in $\mathfrak{M}$ satisfy always the following relation:

$$(\mathrm{A}_2) \qquad \begin{vmatrix} \psi_0^* & u_{01}^n & u_{01}^{n-1}u_{02} & \cdots & u_{02}^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \psi_{n+1}^* & u_{n+11}^n & u_{n+11}^{n-1}u_{n+12} & \cdots & u_{n+12}^n \end{vmatrix} = 0.$$

It follows that there exist in an unique way single valued operations

$$A_{\phi_1, \phi_2}^{n_1, n_2} \quad (n_1, n_2 = 0, 1, \cdots, n;\ n_1 + n_2 = n)$$

on two arguments ϕ_1, ϕ_2 from $\mathfrak{M}$ to $\mathfrak{M}^*$ such that identically

$$(B_2) \qquad A(u_1\phi_1 + u_2\phi_2) = \sum_{\substack{n_1, n_2 = 0 \\ n_1 + n_2 = n}}^{n} \frac{n!}{n_1!\, n_2!} u_1^{n_1} u_2^{n_2} A_{\phi_1, \phi_2}^{n_1, n_2}.$$

Conversely, the existence of such operations with the identity (B_2) implies that relation (A_2) holds. Further similar identities (A_s), (B_s) hold for every s-ary linear system $(\phi_1, \cdots, \phi_s)$.

The operations entering these identities are simply expressible in terms of the operation

$$A_{\phi_1, \cdots, \phi_n}^{1, \cdots, 1}$$

on n arguments occurring in (B_n); this operation is symmetric on its n arguments and linear distributive on each argument, and

$$A(\phi) = A^{1, \cdots, 1}_{\phi_1, \cdots, \phi_n}.$$

Conversely, if $A(\phi_1, \cdots, \phi_n)$ is an operation symmetric on its n arguments and linear distributive on each, then

$$A(\phi) = A(\phi, \cdots, \phi)$$

is an operation on ϕ homogeneous distributive of degree n.

11. Recently Professor Ugo Amaldi has published in the *Memorie* of the Turin Academy a paper on the continuous groups of real conformal transformations in ordinary space. The groups were investigated by the methods of Lie and expressed in the usual infinitesimal notation. In Professor Newson's paper these real conformal transformations are studied by pure geometric methods and all types of such transformations are found. The subgroups of the real conformal group are determined and classified according to their types. Most of Professor Newson's results have been in his possession about eight years, but the investigation was completed only recently. The paper has been accepted for publication by the *Giornale di Matematiche*.

12. The first part of Professor Miller's paper is devoted to a proof of the theorem that a group is necessarily solvable if every non-abelian subgroup is dihedral. This theorem is then extended so as to read: if every non-abelian subgroup of a group is either dihedral or associate dihedral, the group is solvable. It is observed that every non-abelian subgroup of a dihedral group is dihedral, and all the non-dihedral groups in which every non-abelian subgroup is dihedral are investigated. It is proved that if such a group is non-abelian it must belong to one of the following five types: (1) the direct product of the octic group and a group of order p, p being any prime number; (2) the direct product of the dihedral group of order $2q$, q being an odd prime, and a group of order p; (3) the direct product of the dihedral group of order $4q$ and a group of order 2; (4) the group obtained by extending the abelian group of order p^2, $p > 2$, and of type (1, 1) by means of an operator of order 2 which transforms each operator of this abelian group into its inverse; (5) the group of order $4q$, q

being prime and $\equiv 1 \bmod 4$, which is contained in the holomorph of the cyclic group of order q. The first three types contain invariant operators while there is no invariant operator besides identity in either of the last two types.

13. Professor Porter's first paper was a pedagogic note in which attention was called to the fact that if, for example, in three dimensions, we take as our element of volume

$$\Delta = \frac{1}{3!}\begin{vmatrix} \Delta_1 x & \Delta_2 x & \Delta_3 x \\ \Delta_1 y & \Delta_2 y & \Delta_3 y \\ \Delta_1 z & \Delta_2 z & \Delta_3 z \end{vmatrix}$$

and make the one to one change of variable $x = x(u, v, w)$, etc., we get

$$(1) \qquad \Delta = \frac{1}{3!}J\cdot\begin{vmatrix} \Delta_1 u & \Delta_2 u & \Delta_3 u \\ \Delta_1 v & \Delta_2 v & \Delta_3 v \\ \Delta_1 w & \Delta_2 w & \Delta_3 w \end{vmatrix}(1+t)$$

$$= \frac{1}{3!}J\Delta'(1+t),$$

where $\lim_{\Delta'=0} t = 0$, provided the jacobian J of the transformation is continuous.

If the jacobian does not change sign, the tetrahedron Δ' will not overlap if the tetrahedra Δ do not by (1). The usual formulas for change of variable in triple integrals follow at once from Duhamel's principle.

14. Professor Porter's second paper contains further results of his research which began in the current number of the *Annals of Mathematics* and will be continued in the same journal.

15. The purpose of Dr. Wernicke's paper is to extend the applicability of Gauss's criterion for the interlocking of curves. In the double integral

$$\frac{1}{4\pi}\int\int S\frac{\rho-\rho'}{r^3}\,d\rho d\rho'$$

S indicates the scalar product of the vectors, the curves C, C' being given by their vector equations $\rho = \rho(t)$, $\rho' = \rho'(t)$; and

r is the length of $\rho - \rho'$. This integral is an integer m whenever C coils m times about C' in a fixed positive direction. Any bilateral simply connected surface bounded by C' is pierced at least $|m|$ times by C. It was shown that when $m = 0$, C may still so coil about C' and about itself that it cannot be "disentangled." A surface bounded by C may be so constructed that C' does not pierce it. On such a surface, however, an auxiliary curve may be drawn so as to interlock with C' in the former way. Both ways of interlocking may occur simultaneously in the case both of the original and auxiliary curves, if the latter be properly chosen; but a repeated application of Gauss's criterion will reveal all these relations. Moreover, for n-space, the criterion may be so modified as to characterize the interlocking of closed k-surfaces with $(n - k - 1)$-surfaces, where $k = 0, 1, \cdots, n - 1$. Also, while the double integral is not immediately applicable to a curve interlocking with itself, or forming a "knot," it may be used to reveal knottedness, by extending it over the given curve and another accompanying it at a sufficiently small vectorial distance.

16. Dr. Westfall showed, by means of a recent theorem of Stekloff, that the fundamental theorem of Fourier's constants,

$$\int_a^b A(x)f^2(x)dx = \sum_1^\infty a_i^2, \qquad a_i = \int_a^b A(x)f(x)\phi_i(x)dx,$$

holds for any limited and integrable function $f(x)$ and Hilbert's characteristic functions $\phi_i(x)$ of a self adjoined differential equation. It follows that, if $A(x)$ does not vanish in (a, b), two limited and integrable functions having the same generalized Fourier's constants differ by an integrably null function in (a, b), and if $\sum_1^\infty a_i\phi_i(x)$ converges to an integrable and finite function in any sub-interval (α, β), or (a, b), it likewise differs from $f(x)$ in that sub-interval by an integrally null function.

17. Professor Moore's note on the Fourier constants of an integrable function appears elsewhere in this number of the BULLETIN.

18. Although linear transformations with a prime modulus p are employed extensively in many branches of mathematics, the invariants of quantics under such transformations seem to have escaped attention. The first steps in the construction of a the-

ory of such invariants are made in the paper by Professor Dickson. The subject is at once seen to be much more difficult than the ordinary algebraic theory. For instance, the terms of an invariant need not be of the same degree nor of constant weight. Again, the annihilators are far more complicated, involving partial derivatives of higher than the first order. Direct computation is necessarily a luxury in this field.

As the fundamental invariants of $ax^2 + 2bxy + cy^2$, when $p > 2$, we may take the discriminant Δ and

$$J = (a^\mu + 1)(c^\mu + 1)\left\{ \sum_{i=0}^{\mu} a^i c^i b^{2\mu - 2i} - 1 \right\},$$

where $\mu = \frac{1}{2}(p - 1)$. Then $\Delta J = 0$ and $(J + 1)^2 = I + 1$, where

$$I = (a^{p-1} - 1)(b^{p-1} - 1)(c^{p-1} - 1).$$

Replacing p by p^n, we obtain results valid for the Galois field of order p^n. Every quantic has an invariant analogous to I, viz., $\pi(a_i^{p^n-1} - 1)$, the product extending over all the coefficients a_i of the quantic.

A cubic form, for prime modulus $p > 3$, has an absolute invariant K of constant degree $p - 1$ containing $2m$ terms, where m is the number of partitions of $p - 1$ into $3l_0 + 2l_1 + l_2$. Let Δ denote -3 times the algebraic discriminant $a_0^2 a_3^2 + \cdots$ of the cubic form (with binomial coefficients). Then $(\Delta^{\frac{1}{2}(p+1)} - \Delta)K = 0$.

As in these illustrations on quadratic and cubic forms, so in general the new invariants have simple relations with the ordinary algebraic invariants. These results will be incorporated in a paper offered to the *Transactions*.

19. Professor Pell showed that any curve of constant torsion may be considered as the locus of the middle points of the chords connecting corresponding points on a minimal curve and a spherical curve, correlated to this minimal curve. This theorem was obtained from Lie's formulas for constructing a minimal surface passing through a given contour, when the directions of the normals along this contour of the minimal surface are given.

20. For any power series $\sum_0^\infty a_n x^n$, with a radius of convergence r, such that both $\lim_{n=\infty} a_n x^n$ becomes definitely infinite and $|a_n^{-1} x^{-n} \sum_0^n a_n x^n| < \mathrm{L}$ while $|x| > r + e$, Professor

Porter showed that there were points on the circle of convergence which are condensation points for the zeros of the polynomial convergents and that in the neighborhood of every point of this circle the set of these polynomials took on values less than any assigned number however small. He also showed that no set of these polynomials remained limited throughout any region lying outside the circle of convergence.

A. S. CHESSIN,
Secretary.

SELECTED TOPICS IN THE THEORY OF BOUNDARY VALUE PROBLEMS OF DIFFERENTIAL EQUATIONS.

AN ABSTRACT OF FOUR LECTURES DELIVERED AT THE NEW HAVEN COLLOQUIUM, SEPTEMBER 5–8, 1906.

BY PROFESSOR MAX MASON.

§ 1.

FUNCTIONAL equations — particularly integral equations — have aroused much interest and activity in recent years. Important contributions to the subject have been made, notably by Volterra, Fredholm and Hilbert, and the results have found application in the field of mathematical physics and differential equations. The equations studied have been for the most part of a type difficult of solution, and the treatment correspondingly complicated.

There is however a type of functional equation whose solution may be obtained in a simple manner. Consider the equation

$$f = g + Sf, \tag{1}$$

where g is a known function, and S a *linear operator*, that is $S(u + v) = Su + Sv$. *The operator S will be called convergent if the infinite series*

$$\phi + S\phi + S^2\phi + S^3\phi + \cdots$$

converges for all functions ϕ which satisfy the conditions of con-

tinuity demanded of g and f, and if the convergence is such that the operation S may be performed on this series term by term.

Suppose S is convergent, and that a solution f of (1) exists. Then

$$\begin{aligned} f &= g + Sf = g + S(g + Sf) \\ &= g + Sg + S^2 f = g + Sg + S^2(g + Sf) \\ &= g + Sg + S\,g + S^3 f = \cdots, \end{aligned}$$

the equation

$$f = g + Sg + S^2 g + S^3 g + \cdots + S^{n-1} g + S^n f$$

being obtained after $n - 1$ substitutions of $g + Sf$ for f. If S is convergent, the limit for $n = \infty$ may be taken. It follows that if a solution of (1) exists it has the form

$$f = g + Sg + S^2 g + S^3 g + \cdots. \tag{2}$$

Conversely, if S is a convergent operator the function defined by (2) is a solution of (1), for

$$\begin{aligned} Sf &= Sg + S^2 g + S^3 g + \cdots \\ &= f - g. \end{aligned}$$

To summarize: *If S is a convergent operator, the equation* (1) *admits a unique solution, given by equation* (2).

A simple example is furnished by the equation

$$f(x) = g(x) + \int_a^x K(x, y) f(y) dy. \tag{3}$$

The operator S is convergent. For if Φ be the maximum of the absolute value of a function ϕ, and κ the maximum of K, then

$$| S^n \phi | \leqq \frac{(x - a)^n \kappa^n}{n!} \Phi.$$

These results will be applied to the study of certain boundary value problems of differential equations. This method has some advantages over the method of successive approximations, which has been used with such success, notably by Picard. The attention is at once directed to the essential difficulty, that of convergence, and the uniqueness of the solution is proved without special investigation. The method is applicable however only to linear equations.

§ 2.

The linear partial differential equation of the second order in two independent variables

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

is said to be of elliptic, hyperbolic, or parabolic type, according as $AC - B^2$ is positive, negative, or zero, in the region considered. The normal forms of these types are

$$u_{xx} + u_{yy} = au_x + bu_y + cu + f,$$

$$u_{xy} = au_x + bu_y + cu + f, \quad u_{xx} = au_x + bu_y + cu + f,$$

the general equation being reduced to the normal form of the type to which it belongs, by a transformation of the independent variables.

The most important representative of the elliptic type is the potential equation

$$u_{xx} + u_{yy} = 0, \tag{4}$$

and the boundary value problems for this equation are fundamental in the treatment of similar problems for the general equation of elliptic type. Dirichlet's problem — to determine a solution of (4) within a closed region Ω which assumes given values on the boundary of Ω — is the most famous of boundary value problems. *There exists a unique solution of this problem.* The proofs of this statement given by Schwarz, Neumann, and Poincaré are classic. In case the region Ω is convex, Neumann's method gives the simplest solution. The results of § 1 may be applied to interpret this method. The function

$$u(x, y) = \frac{1}{\pi}\int_0^l f(t) \frac{\partial}{\partial t} \Theta[x, y, \xi(t), \eta(t)]dt, \quad \Theta = \arctan\frac{\eta(t) - y}{\xi(t) - x},$$

containing the arbitrary function $f(t)$, is a solution of (4) at all points within Ω, the boundary of Ω being given by the curve

$x = \xi(t), y = \eta(t)$. This function $f(t)$ must be so determined that u takes the value $g(s)$ at the point $x = \xi(s)$, $y = \eta(s)$. On account of the discontinuity of u in crossing the boundary this leads to the following integral equations for f:

$$(5) \qquad f(s) = g(s) - \frac{1}{\pi}\int_0^l f(t)\frac{\partial\Theta(s, t)}{\partial t}\,dt,$$

where $\Theta(s, t) = \Theta(\xi(s), \eta(s), \xi(t), \eta(t))$. This equation is of the type considered in § 1, but the operator is not convergent. However, on adding $f(s)$ to both sides and dividing by two, the equation takes the form

$$f(s) = \tfrac{1}{2}g(s) + Sf,$$

where

$$Sf = -\frac{1}{2\pi}\int_0^l \{f(t) - f(s)\}\,\frac{\partial\Theta(s, t)}{\partial t}\,dt.$$

The convergence proof of Neumann's method shows that the operator S is convergent. Hence a unique solution of (5) exists and is given by

$$f = \tfrac{1}{2}\{g + Sg + S^2g + \cdots\}.$$

The function u produced by this function f is the desired solution of Dirichlet's problem.

On account of the above existence theorem, the existence of a Green's function is assured. This function has the form

$$G(x, y, \xi, \eta) = \log\frac{1}{\sqrt{(x - \xi)^2 + (y - \eta)^2}} + g(x, y, \xi, \eta),$$

where (ξ, η) is a parameter point within Ω, and G vanishes identically in ξ, η when the point (x, y) lies on the boundary of Ω. By the aid of this function a solution of the equation

$$(6) \qquad u_{xx} + u_{yy} = f(x, y)$$

which assumes the values $g(s)$ on the boundary of Ω is given by the formula

$$(7) \qquad u = u_0 - \frac{1}{2\pi}\int\int_\Omega G(x, y, \xi, \eta)f(\xi, \eta)\,d\xi d\eta,$$

where u_0 is the solution of Dirichlet's problem for the boundary values $g(s)$, provided that f satisfies a certain continuity condition.

A more general equation of elliptic type, and one of frequent occurrence in the applications, is

$$(8) \qquad v_{xx} + v_{yy} = cv + f.$$

Replacing f in equation (6) by $cv + f$ it is seen that the solution of (8) which assumes the boundary values $g(s)$ is given by the solution of the integral equation

$$(9) \qquad v = u - \frac{1}{2\pi}\int\int_{\Omega} G(x, y, \xi, \eta)c(\xi, \eta)v(\xi, \eta)d\xi d\eta,$$

where u is a known function, given by equation (7). It may be easily shown that the operator S in this equation is convergent, if the region Ω be sufficiently small, the proof depending on the fact that

$$\int\int_{\Omega} G d\xi d\eta$$

approaches zero with the area of Ω. *There exists one and only one solution of the boundary value problem of equation* (8) *if the region* Ω *is sufficiently small.*

The general equation of elliptic type,

$$(10) \qquad u_{xx} + v_{yy} = av_x + bv_y + cv + f$$

presents greater difficulties. To determine a solution of the boundary value problem, a functional equation of the form

$$(11) \qquad v = u - \frac{1}{2\pi}\int\int_{\Omega} G(av_x + bv_y + cv)d\xi d\eta$$

is to be solved. The operator S involves differentiation as well as integration. The proof that S is convergent is accordingly more difficult, and an investigation regarding continuity is necessary to show that a solution of (11) is also a solution of (10); the results are however the same as in the case of equation (8). This existence theorem was first obtained by Picard, who used the method of successive approximations. The restriction on the size of the region in the theorem is not one due to method alone. In point of fact, if the region be allowed to increase in size, a position is reached for which the theorem is no longer true. Some of the most interesting investigations of special boundary value problems deal with the equation

$$v_{xx} + v_{yy} = \lambda cv,$$

where λ is a parameter, the region Ω being fixed. The results are analogous to those obtained in § 4 for a similar problem in ordinary differential equations.

To Picard is due a remarkable theorem regarding the nature of solutions of the differential equation (10), viz.: *Every equation of elliptic type whose coefficients are analytic functions possesses only analytic solutions*; even though the solution assumes non-analytic boundary values, it will be analytic inside the region.

§ 3.

The equations of hyperbolic type are especially susceptible to treatment by the method of § 1. By the aid of this method a boundary problem may be solved which includes as special cases many problems whose solutions were originally obtained by quite distinct methods.

The equation

$$(12) \qquad u_{xy} = au_x + bu_y + cu + f$$

is to be solved under the conditions

$$(u)_{y=\phi(x)} = U(x), \qquad (u_y)_{x=\psi(y)} = Y(y),$$

where ϕ, ψ, U, Y, are given functions.

Let $u_x = v$. The function

$$u = \int_{\phi(x)}^{y}\int_{\psi(\eta)}^{x} v(\xi, \eta)d\xi d\eta + \int_{\phi(x)}^{y} Y(\eta)dy + U(x)$$

satisfies the boundary conditions, whatever v be. On substituting this value u in (12), the equation becomes a functional equation for the determination of v, of the form

$$v = g + Sv,$$

where S is of somewhat complicated form, but involves integration only, the upper limits of each integral being x or y. On account of this fact the proof that S is a convergent operator may be made without difficulty, however large the region within which the point (x, y) may vary. The proof is analogous to that given in connection with equation (3). *It follows that there exists one and only one solution of the boundary problem.* If ϕ and ψ are inverse functions, so that $x = \psi(y)$ and $y = \phi(x)$ define the same curve, the problem is equivalent to finding a solution u of (12) when u and the normal derivative of u are given on a monotonic curve. If $\phi(x) = b$, $\psi(y) = a$, the boundary condition is equivalent to giving the values of u on the lines $x = a$, $y = b$. Other special cases may be easily obtained.

It may be readily seen from the expression for u in terms of v that the continuity properties of the functions ϕ, ψ, U, Y determine to a large extent the continuity properties of u throughout its region of definition. The equations of hyperbolic type thus differ fundamentally from the equations of elliptic type. In fact, S. Bernstein has recently proved that every equation of hyperbolic type possesses non-analytic solutions.

The general equation of parabolic type offers exceptional difficulties. But little is known of the nature of the solutions or of the boundary conditions which may be placed upon them. These equations form a limiting case between those of elliptic and hyperbolic types. Practically nothing is known regarding the boundary problems of equations which are of one type in a part of the region considered and of another in the remainder.

§ 4.

Many of the boundary value problems for partial differential equations may be reduced by the familiar substitution $u = X(x) \cdot Y(y)$, to boundary value problems for ordinary differential equations. The differential equation

$$y'' + \lambda A(x) y = 0, \tag{13}$$

where λ is a parameter, is typical of a large class, and has formed the subject of numerous investigations. The fundamental existence theorem for this equation may be easily proved by the method of § 1. The function

$$y = \int_a^x (x - \xi) u(\xi) d\xi + \alpha'(x - a) + \alpha$$

satisfies the conditions $y(a) = \alpha$, $y'(a) = \alpha'$, whatever be u. It is a solution of (13) if u is a solution of the integral equation

$$u = -\lambda A(x)[\alpha'(x-a)+\alpha] - \lambda A(x)\int_a^x (x-\xi)u(\xi)d\xi,$$

which is a special case of (3). There exists accordingly one and only one solution of (13) satisfying the initial conditions. From the form of the solution it is seen immediately that y is an integral transcendental function of the parameter λ.

A typical boundary problem for (13) is to determine a solution, not identically zero, which vanishes for two given values a and b of x. The general solution of (13) having the form

$$y = c_1\eta_1 + c_2\eta_2,$$

this problem may be solved when and only when λ is a zero of the transcendental function

$$\delta(\lambda) = \begin{vmatrix} \eta_1(a) & \eta_2(a) \\ \eta_1(b) & \eta_2(b) \end{vmatrix}.$$

This problem was first studied by Sturm, and his results may be summarized in the following theorem — Sturm's theorem of oscillation: *If $A(x)$ does not change sign the parameter λ may be determined in one and only one way so that a solution of (13) exists which vanishes at a and at b, and n times between a and b.*

This result has since been proved by various methods, some of which are capable of greater generalization than the original methods of Sturm. The problem may be reduced to the solution of an integral equation. Any solution of the equation

$$y(x) = \lambda\int_a^b A(\xi)G(x,\xi)y(\xi)d\xi,$$

where

$$G(x,\xi) = \tfrac{1}{2}\left\{|x-\xi| - \frac{(b-x)(\xi-a)+(b-\xi)(x-a)}{b-a}\right\},$$

is a solution of (13) and vanishes at a and b. This integral equation is of a type studied by Fredholm and Hilbert. If

$A(x)$ does not change sign, it can be reduced to the form

$$\eta(x) = \lambda \int_a^b K(x, \xi)\eta(\xi)d\xi,$$

where K is symmetric, the only case to which Hilbert's results apply.* The application of these results proves immediately the existence of an infinite series of "normal parameter values" — zeros of $\delta(\lambda)$ — for each of which there exists a "normal function," a solution of the boundary value problem. Hilbert proves further that any function may be expanded in terms of these normal functions, if it is continuous together with its first and second derivatives. The most general result regarding the expansion of an arbitrary function have however been attained by Kneser, using a different method.

If the function $A(x)$ changes sign, the above method does not apply. The existence theorem may however be obtained by a method based on the consideration of certain minimum problems. Interesting properties of the normal functions are thus set in evidence, and by the aid of these properties a very simple proof for the expansion of a function in terms of normal functions is obtained, even in the case that $A(x)$ changes sign.

The field of boundary value problems is of enormous extent, and comparatively little is known territory. While the theorems stated above are typical, interesting variations result by generalization to more variables, higher derivatives, and more general boundary conditions. Some of these generalizations have been rigorously made, many merely guessed. In particular, the boundary value problems for systems of differential equations offer a wide field for research, and work in this field would be of especial value in the applications.

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* Since the above was written a fifth installment of Hilbert's Grundzuge einer allgemeinen Theorie der linearen Integralgleichungen has appeared, in which the case where the function K is not symmetric is considered. (*Göttinger Nachrichten*, 1906, p. 439.)

NOTE ON FOURIER'S CONSTANTS.

BY PROFESSOR E. H. MOORE

(Read at the Preliminary Meeting of the Southwestern Section of the American Mathematical Society, December 1, 1906.)

A FUNCTION $f(x)$ real and single valued, bounded, and in the sense of Riemann integrable from 0 to 2π gives rise to the Fourier's series

$$(1)\quad \tfrac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) = \tfrac{1}{2}\sum_{k=-\infty}^{+\infty}(a_k \cos kx + b_k \sin kx),$$

where

$$(2)\quad \begin{aligned} a_k &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx, \\ b_k &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx. \end{aligned} \qquad (k = 0, \pm 1, \cdots)$$

Hurwitz[*] calls the constants a_k, b_k the Fourier's constants of the integrable function $f(x)$, and for the realm of integrable functions he studies the theory of Fourier's constants instead of the theory of Fourier's series which are convergent only under conditions.

The realm of integrable functions is a realm of integrity, *i. e.*, the sum, the difference, and the product of two integrable functions is integrable. The constants a_k, b_k, or in more explicit notation a_{kf}, b_{kf}, depend in a linear distributive way on the function $f = f(x)$. Further, the constants a_{kh}, b_{kh}, for the product $h(x) = f(x)\,g(x)$ of two functions $f(x)$, $g(x)$ are determinable from the constants a_{kf}, b_{kf} and a_{kg}, b_{kg} of these functions. In fact, from (2) we have

$$(3)\quad a_{0f} = \frac{1}{\pi}\int_0^{2\pi} f(x)dx,$$

$$(4)\quad a_{kf} = a_{0,f(x)\cos kx}, \qquad b_{kf} = a_{0,f(x)\sin kx},$$

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[*] Hurwitz, "Ueber die Fourierschen Konstanten integrierbarer Funktionen," *Math. Annalen*, vol. 57 (1903) pp. 425–446, and vol. 59 (1904), p. 553.

so that

$$(5) \qquad a_{kh} = a_{0,fg\cos kx}, \qquad b_{kh} = a_{0,fg\sin kx}.$$

These formulas in effect make the expression of a_{kh}, b_{kh} depend simply upon the expression of a_{0h} in terms of the constants of f and g. The fundamental theorem of the theory of Fourier's constants states that

$$(6) \qquad a_{0h} = \frac{1}{\pi}\int_0^{2\pi} f(x)g(x)dx = \tfrac{1}{2}a_{0f}a_{0g} + \sum_{k=1}^{\infty}(a_{kf}a_{kg} + b_{kf}b_{kg}).$$

If either $f(x)$ or $g(x)$ is expressible as a uniformly convergent Fourier series, the relation (6) follows by termwise integration. In its generality the fundamental theorem with important applications is due to de la Vallée Poussin * (1893), and it has been extended (1903) by Stekloff to all the types of Fourier constants frequently employed in analysis in connection with the representation of arbitrary functions of one or of several variables.

Hurwitz (loc. cit., volume 57, page 437) expresses the constants of $h(x) = f(x)\,g(x)$ in terms of those of $f(x)$, $g(x)$, apart from notation, as follows:

$$a_{0h} = \tfrac{1}{2}a_{0f}a_{0g} + \sum_{k=1}^{\infty}(a_{kf}a_{kg} + b_{kf}b_{kg}),$$

$$(7) \quad a_{nh} = \tfrac{1}{2}a_{0f}a_{ng} + \tfrac{1}{2}\sum_{k=1}^{\infty}\{a_{kf}(a_{k+n\,g}+a_{k-n\,g}) + b_{kf}(b_{k+n\,g}+b_{k-n\,g})\},$$

$$b_{nh} = \tfrac{1}{2}a_{0f}b_{ng} + \tfrac{1}{2}\sum_{k=1}^{\infty}\{a_{kf}(b_{k+n\,g}-b_{k-n\,g}) - b_{kf}(a_{k+n\,g}-a_{k-n\,g})\}.$$

Hurwitz remarks further that these series converge absolutely, even when the terms at present collected into binomial expressions are taken as the constituent terms of the series. In the formulas (7) the rôles of f and g may of course be interchanged.

The series on the right of the equations (7) should however be expressed symmetrically with respect to f and g.

Let us use as Fourier constants

$$(2) \quad A_{kf} = \frac{1}{2\pi}\int_0^{2\pi} f(x)\cos kx\,dx, \qquad B_{kf} = \frac{1}{2\pi}\int_0^{2\pi} f(x)\sin kx\,dx,$$

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* For references see the memoirs of Hurwitz

viz., the halves of the constants according to Hurwitz. Then with use of the relations

$$(\bar{2}^\circ) \qquad A_{-kf} = A_{kf}, \qquad B_{-kf} = -B_{kf}$$

one readily finds from (7)

$$(\bar{7}) \qquad \begin{aligned} A_{nh} &= \sum_{k+l=n}^{k,\,l=0,\,\pm 1,\,\dots} (A_{kf}A_{lg} - B_{kf}B_{lg}), \\ B_{nh} &= \sum_{k+l=n}^{k,\,l=0,\,\pm 1,\,\dots} (A_{kf}B_{lg} + B_{kf}A_{lg}). \end{aligned} \qquad (n = 0, \pm 1, \cdots)$$

These equations $(\bar{7})$ have the desired symmetric form. The form suggests moreover the introduction of the complex Fourier constants

$$(\bar{2}_a) \qquad C_{kf} = A_{kf} + jB_{kf} \qquad (j^2 = -1),$$

in terms of which we have

$$(\bar{7}_a) \qquad C_{nh} = \sum_{k+l=n}^{k,\,l=0,\,\pm 1,\,\dots} C_{kf}C_{lg} \quad (n = 0, \pm 1, \cdots).$$

This result may be expressed thus:

To every integrable function $f(x)$ *there corresponds a system of Fourier constants* A_{kf}, B_{kf}, C_{kf} $(2, \bar{2}_a)$ *and a Laurent series with coefficients* C_{kf}

$$f(x) \sim \sum_{k=-\infty}^{+\infty} C_{kf} Z^k.$$

This Laurent series is to be considered formally. Then the Laurent series corresponding to the product of two functions is the formal product of the two Laurent series corresponding respectively to the two functions.

As thus expressed, the fundamental theorem of the theory of Fourier's constants points to the known expression of Fourier series for suitably conditioned functions as convergent Laurent series.

THE UNIVERSITY OF CHICAGO,
December 7, 1906.

ON THE MINIMUM NUMBER OF OPERATORS WHOSE ORDERS EXCEED TWO IN ANY FINITE GROUP.

BY PROFESSOR G. A. MILLER.

LET the order of any group G be represented by $g = 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\lambda^{\alpha_\lambda}$; $p_1, p_2, \cdots, p_\lambda$ being distinct odd prime numbers. When $\alpha_0 = 0$ all the operators of G except identity are of orders which exceed two. Hence we shall assume in what follows that $\alpha_0 > 0$. In this case it is easy to see that there is at least one group of order g in which the number of operators whose orders exceed two is exactly

$$\frac{(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\lambda^{\alpha_\lambda} - 1)g}{2 p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\lambda^{\alpha_\lambda}} = N.$$

Such a group may be constructed by forming the direct product of a group of order $2 p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\lambda^{\alpha_\lambda}$ in which just half the operators are of order two * and the abelian group of order $2^{\alpha_0 - 1}$ and of type $(1, 1, 1, \cdots)$. The main objects of this paper are to prove that N is the minimum number of operators whose orders exceed two that can occur in a group of order g, and if a group of this order has exactly N operators whose orders exceed two it is the direct product of a group in which just half the operators are of order two and the abelian group of order $2^{\alpha_0 - 1}$ and of type $(1, 1, 1, \cdots)$.

When $g = 2^{\alpha_0}$, $N = 0$ and the theorem requires no proof. When $\alpha_0 = 1$, $N = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\lambda^{\alpha_\lambda} - 1$ and the theorem is evident from the well-known theorem that every group whose order is twice an odd number contains a subgroup of half its order which is composed of operators of odd order in addition to identity. Hence we shall assume in what follows that $\alpha_0 > 1$ and $N > 0$. Since G is supposed to contain a minimum number of operators whose orders exceed two, it may be assumed that over half of the operators of G are of order two. Hence G is generated by its operators of order two, and as it contains oper-

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* If just half the operators of a group are of order 2, the order of the group is twice an odd number, and all its operators of odd order together with identity constitute an abelian subgroup whose order is half the order of the group.

ators whose orders exceed two, it must be non-abelian. It must, therefore, contain non-invariant operators of order two.

Let H_1 be the subgroup of G which is composed of all the operators of G which are commutative with one of its non-invariant operators t_1 of order two. The products obtained by multiplying t_1 into operators of order two in $G - H_1$ are of orders which exceed two, since all of these operators of order two are non-commutative with t_1. Hence $G - H_1$ contains at least as many operators whose orders exceed two as of order two. From this it follows that more than half the operators of H_1 are of order two. If H_1 contains operators whose orders exceed two it is non-abelian and vice versa.

When H_1 is non-abelian, it contains a non-invariant operator t_2 of order two. In this case its subgroup composed of its operators which are commutative with t_2 may be denoted by H_2. More than half the operators of H_2 are of order two; and, if it is non-abelian, we continue in the same way until we arrive at an abelian subgroup H_m. All the operators of H_m except identity are of order two. Since at least half of the operators of $H_x - H_{x+1}$ $(x = 1, 2, \cdots, m - 1)$ are of orders which exceed two, it follows that at least half the operators of $G - H_m$ are of orders which exceed two.

It is possible that different choices of non-invariant operators of order two might affect the order of H_m. We suppose that the operations were effected in such a manner that the order of H_m is as large as possible. Such an H_m can be obtained by a finite number of operations since g is finite. The index of H_m under G cannot be less than $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_\lambda^{\alpha_\lambda}$, as the order of H_m is a power of 2. Since at least half of the operators in $H_x - H_{x+1}$ are of orders which exceed two, it is clear that the minimum number of operators whose orders exceed two in a group of order g is one-half the number of operators in $G - H_m$ when the index of H_m under G has its minimum value, viz., $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_\lambda^{\alpha_\lambda}$. That is, *$N$ is the minimum number of operators whose orders exceed two in any group of order g.*

It remains only to determine all the possible groups of order g which contain exactly N operators whose orders exceed 2. In what follows it will be assumed that G satisfies this condition. If an operator of order two in $H_{m-1} - H_m$ were commutative with some operator whose order exceeds two in H_{m-1} their product would be in $H_{m-1} - H_m$ and would be transformed into its inverse by t_m. As t_m would transform into its inverse the given operator whose order exceeds two as well as this product, it would have to be commutative with the operator of order two from $H_{m-1} - H_m$. Since this is impossible, it follows that all the operators of order two in $H_{m-1} - H_m$ are commutative with exactly 2^{α_0} operators in H_{m-1}* and that these operators constitute a subgroup which is conjugate with H_m under H_{m-1}.

Since all the subgroups of order 2^{α_0} are conjugate under H_{m-1} and one of them contains an operator which is non-invariant under H_{m-1}, all of these Sylow subgroups must have this property. If an operator occurs in two of them it must occur in all of them, since it is commutative with operators of H_{m-1} whose orders exceed two. All the non-invariant operators of order two in H_{m-1} must therefore transform every operator whose order exceeds two in H_{m-1} into its inverse. As one of the latter operators is transformed into its inverse by at least half the operators in $H_{m-1} - H_m$ and by the operators of H_m which are not commutative with every operator of H_{m-1}, it must be transformed into its inverse by just half the operators of H_{m-1}. Hence H_m contains just 2^{α_0-1} operators which are invariant under H_{m-1}.

There are as many operators in H_{m-1} that are commutative with one of its operators whose order exceeds two as there are of those which transform this operator into its inverse. As the latter includes half the operators of H_{m-1} the former set must be composed of the remaining half. Hence every operator whose order exceeds two in H_{m-1} is commutative with each of the other operators in H_{m-1} which have this property. From this it follows that all the operators of odd order in H_{m-1} together with identity constitute an abelian subgroup, and that any non-invariant operator of order two together with this abelian subgroup generate an invariant subgroup in which just half its operators are of order two. This subgroup is invariant since it includes all the conjugates of its operators of order two. Hence H_{m-1} *is the direct product of a subgroup in which just half the operators are of order two and the subgroup of order* 2^{α_0-1} *formed by the invariant operators of* H_{m-1}.

We shall now prove that the order of H_α is divisible by $p_1^{\alpha_1}$ whenever it is divisible by p_1. Suppose that $p_1^{\alpha_1'}$ is the highest power of p_1 which divides the order of H_x and that the order of H_{x-1} is divisible by $p_1^{a'+1}$. Hence H_{x-1} contains a subgroup of order $p_1^{a'+1}$ which has $p_1^{a'}$ operators in H_x and the remaining operators in $H_{x-1} - H_x$. All the operators of H_x are commutative with t_x and all the operators whose orders exceed two in $H_{x-1} - H_x$ are transformed into their inverse by t_x. As this is impossible* it follows that the order of H_x is divisible by $p_1^{a_1}$ whenever it is divisible by p_1.

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* Every operator of order two must be found in one of the Sylow subgroups of order 2^{α_0}, and all of these subgroups are similar to H_m.

Our next object is to prove that H_{m-1} is identical with G. This will be proved by showing that the opposite hypothesis leads to an absurdity. Suppose that the order of H_{m-1} is $2^{a_0} p_1^{a_1} \cdots p_{\lambda'}^{a_{\lambda'}}$, $\lambda' < \lambda$. Hence the order of H_{m-2} is divisible by primes which do not divide the order of H_{m-1}. The operator t_{m-1} of order two which is invariant under H_{m-1} but not under H_{m-2} is found in the subgroup K of H_{m-1} which is composed of its 2^{a_0-1} invariant operators. All the operators whose orders exceed two in $H_{m-2} - H_{m-1}$ are transformed into their inverses by t_{m-1}. As the number of conjugates of t_{m-1} under H_{m-2} is the index of H_{m-1} under H_{m-2} it follows that the operators in $H_{m-2} - H_{m-1}$ cannot be transformed into more different operators to obtain their inverses than the given index. Hence these operators cannot include more operators of odd order than this index, since two distinct operators of odd order must be multiplied by two distinct operators in order to obtain products which are the inverses of the first two operators.

It is now easy to prove that the Sylow subgroup of order $p_1^{a_1}$ contained in H_{m-1} is transformed into itself by operators of odd order in $H_{m-2} - H_{m-1}$. This is true when this is the only subgroup of order p^{a_1} in H_{m-2}. It is also true when this Sylow subgroup has less conjugates under H_{m-2} than the index of H_{m-1} under H_{m-2}, since this index involves no primes which divide the order of H_{m-1}. The remaining cases are when this Sylow subgroup has as many conjugates under H_{m-2} as this index and some of these Sylow subgroups have common operators or no two of them have common operators. The former of these two cases is excluded by the last footnote, since the Sylow subgroup in H_{m-1} would have operators in common with some other Sylow subgroup. The latter of the two cases is excluded by the fact that the number of operators of odd order in $H_{m-2} - H_{m-1}$ cannot exceed the index of H_{m-1} under H_{m-2}.

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*The truth of this statement follows from the elementary theorem: If an operator transforms all the operators of a group which are not included in an invariant subgroup of index $k > 2$ into their inverses it transforms every operator of the group into its inverse and the group is abelian. Hence all the Sylow subgroups of odd order contained in G are abelian.

Having proved that the assumption that the order of H_{m-1} is less than the order of G requires that the Sylow subgroup of order $p_1^{a_1}$ in H_{m-1} is transformed into itself by operators of odd order in $H_{m-2} - H_{m-1}$, it is now easy to see that this assumption leads to an absurdity. In fact, the theorem of the last footnote shows clearly that this subgroup of order $p_1^{a_1}$ cannot be transformed into itself by any operator of odd order in $H_{m-2} - H_{m-1}$, since it would be invariant in a group which would have only $p_1^{a_1}$ operators in common with H_{m-1} and the remaining operators of this group would be transformed into their inverses by t_{m-1}. Hence the theorem: *If a group of order g contains the smallest possible number of operators whose orders exceed two, the subgroup which is composed of all its operators which are commutative with one of the non-invariant operators of order two contains no operator whose order exceeds two.* Moreover, this subgroup is a Sylow subgroup, and just half of the remaining operators are of order two.

If we combine with this theorem the one stated above in regard to H_{m-1}, we have a complete determination of all the possible groups of order g in which the number of operators whose orders exceeds two is a minimum. To construct these groups it is only necessary to construct all the possible abelian groups of order $p_1^{a_1}p_2^{a_2}\cdots p_\lambda^{a_\lambda}$ and add to each of them an operator of order two which transforms each of its operators into its inverse. The groups thus obtained may be multiplied directly into the group of order 2^{a_0-1} which contains only operators of order two in addition to identity. Hence there are as many such groups as there are abelian groups of order $p_1^{a_1}p_2^{a_2}\cdots p_\lambda^{a_\lambda}$.

The preceding considerations show that the minimum number of operators whose orders exceed two in any group with the single exception of the abelian group of order 2^a and of type $(1, 1, 1, \cdots)$ is one fourth of the order of the group. If more than one fourth of the operators have orders which exceed two, at least one third of the operators have this property. When the order of the group is a power of 2, all the possible ratios between the number of the operators whose orders exceed two and the order of the group have been determined on the hypothesis that this ratio is less than one half. It is clear that N, in the present article, has for its limits one third and one half respectively, — the former being attained, while the latter is not attained.

NOTE ON THE ORIENTATION OF A SECANT.

BY PROFESSOR L. D. AMES.

(Read at the Preliminary Meeting of the Southwestern Section of the American Mathematical Society, December 1, 1906.)

IN a previous paper* the orientation of curves and their tangents is briefly discussed. For certain problems in the theory of the potential function it is desirable to be able to define uniquely the positive sense along a secant through two variable points of an open smooth curve in such a way that the angle α between the positive direction of the secant and a fixed line shall be a continuous function of the two variable points. Such a definition will include the tangent as a special case. A natural method would be to define the positive sense uniquely and then prove that α is continuous. Another method would be to define the positive sense of the secant for one position and then define the positive sense in every other position so that the angle α shall be continuous. Either method involves somewhat troublesome considerations if the angle is defined by means of any one of the inverse trigonometric functions. The method of defining an angle used in the paper above mentioned proves useful in this case.

Given the open curve

$$x = x(t), \quad y = y(t), \qquad t_0 \leqq t \leqq T,$$

where x, y, dx/dt, dy/dt are single-valued and continuous in the interval (t_0, T), and where

$$x(t_1) \neq x(t_2), \quad y(t_1) \neq y(t_2) \quad \text{when } t_1 \neq t_2.$$

Define a secant through the points t_1, t_2 as follows:

$$X = \epsilon\lambda\rho + x(t_1), \quad Y = \epsilon\mu\rho + y(t_1),$$

where ϵ is arbitrarily chosen either $+1$ or -1, ρ is the variable parameter, and

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* "An arithmetic treatment of some problems in analysis situs," *Amer. Jour. of Mathematics*, Oct., 1905, p. 349.

$$\lambda = \frac{x(t_2) - x(t_1)}{t_2 - t_1}, \qquad \mu = \frac{y(t_2) - y(t_1)}{t_2 - t_1} \quad \text{when} \quad t_1 \neq t_2$$

and

$$\lambda = \frac{dx_1}{dt}, \qquad \mu = \frac{dy_1}{dt}, \qquad \text{when} \quad t_1 = t_2.$$

Define the *positive sense* along this line to be the direction of increasing ρ. This is uniquely defined for any pair of points in the interval (t_0, T) since λ and μ are unchanged by interchanging t_1 and t_2.

Define an angle α as follows:

$$\sin\alpha = \epsilon k\mu, \quad \cos\alpha = \epsilon k\lambda, \quad k = (\lambda^2 + \mu^2)^{-\frac{1}{2}}.$$

Then α is an infinitely many-valued function of t_1 and t_2, its values for any given pair of values of t_1 and t_2 differing by multiples of 2π. If one of these values α' be assigned to a particular pair t_1', t_2', then from the possible values of α one and only one single valued continuous function can be chosen which takes the value α' at t_1', t_2'.*

UNIVERSITY OF MISSOURI,
November 5, 1906.

ON EULER'S ϕ-FUNCTION.

BY PROFESSOR R. D. CARMICHAEL.

(Read before the American Mathematical Society, December 28, 1906.)

THE object of the present note is the demonstration of certain very elementary propositions concerning Euler's ϕ-function of a number.

I. *The relation $\phi(m) = n$, a given number, is never uniquely satisfied for any given value of n.* That is, there is always more than one value of m for every possible value of n.

If any solution is $m =$ an odd number, then the given relation is satisfied by $2m$ also. Likewise, if m is twice an odd number, we may show that $m/2$ will also satisfy the relation.

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* Cf., e. g., Stolz, Differential-Rechnung, vol. 2, p. 15–20.

Hence, if there is a unique solution, m is a multiple of 4; say $m = 4\mu$. Now n is even; say $n = 2\nu$. Then we have

$$\phi(4\mu) = 2\nu.$$

Hence

$$\phi(2\mu) = \nu.$$

Then in a manner similar to the above we may show that μ and ν are both even. By continuing the process step by step we are able to show that a unique solution cannot exist unless both m and n are powers of 2. It remains therefore to show that this cannot give a unique solution. Let $n = 2^{\alpha}$. Then

$$\phi(m) = 2^{\alpha}$$

is satisfied not only by $m = 2^{\alpha+1}$ but also by $m = 2^a(2^b + 1)(2^c + 1) \cdots$ in every way in which $a, b, c, \cdots$ can be so chosen that $a + b + c + \cdots = \alpha + 1$, $a \neq 0$; or $b + c \cdots = \alpha$, $a = 0$; and $2^b + 1$, $2^c + 1, \cdots$ shall be different primes. If $\alpha \geqq 3$, one such solution is always $a = \alpha - 2$, $b = 1$, $c = 2$. An examination for the smaller values of α shows that no unique solution exists in these cases. Hence the proposition

II. *The equation* $\phi(m) = 2^n$ *has just* $n + 2$ *solutions when* $n + 2 \leqq 33$; *but just* 33 *solutions for* $n =$ *any number from* 32 *to* 255.

An odd solution evidently requires

$$m = (2^{\alpha} + 1)(2^{\beta} + 1)(2^{\gamma} + 1) \cdots,$$

where $\alpha + \beta + \gamma + \cdots = n$ and each factor is prime. Since 2^x+1 is prime* for $x = 1, 2, 4, 8, 16$, but not for $x = 32, 64, 128$ nor any value of x not of the form 2^p, it is clear that $\phi(m) = 2^n$ has one and but one odd solution for every value of n up to $n = 31$. Also twice every even value of m which satisfies $\phi(m) = 2^n$ will satisfy the equation when the second member becomes 2^{n+1}. Hence the number of solutions is increased by one when n is so increased up to $n = 31$; but beyond that up to $n = 255$ the number of solutions remains constant; for there is then no solution except those given by twice each solution for the preceding value of n. Up to $n = 31$ it is easy to see that the number of solutions is $n + 2$; then from this point onward to $n = 255$ the number remains constant and is 33.

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*See BULLETIN, June 1906, p. 449.

It will be noticed that the first value of 2^n to which there corresponds no odd solution in m is 2^{32}. This is the smallest value of $\phi(m)$ known to the writer to have no odd solution in m.

III. COROLLARY. *The equation* $\phi(m) = 2^n$ *has (only) one odd solution when* $n \leqq 31$; *otherwise no odd solution at all up to* $n = 255$. *Also it has evidently no other odd solution except for such values of* n *as make* $2^n + 1$ *prime.*

IV. *All the solutions of the equation* $\phi(m) = 4n - 2$, $n \neq 1$, *are of the form* p^a *and* $2p^a$, *where* p *is a prime of the form* $4s - 1$.

Now $m \neq 4$. Then if m contains the factor 4 it is evident that the equation is not satisfied. Neither is it satisfied if m contains two odd primes. Therefore the only values left are of the form p^a and $2p^a$. Moreover p must be a prime of the form $4s - 1$; for otherwise the equation is not satisfied. (There may evidently be more than one p which furnishes such a solution. A case in point is $\phi(m) = 18$, which has the solutions $m = 19$, 27, 38, 54.)

V. *If* p *is of the form* $4s - 1$ *and* $\phi(m) = p^a(p - 1)$ *has but the two solutions* $m = p^{a+1}$, $2p^{a+1}$, *then the relation* $\phi(m) = 2p^a(p - 1)$ *has an odd solution.* (a belongs to the series 0, 1, 2, $\cdots$.)

For one solution of the latter is $m = 4p^{a+1}$. There is no other solution in which m is a multiple of 4; for then there would correspond to that a third solution for $\phi(m) = p^a(p - 1)$. But $\phi(m) = 2p^a(p - 1)$, by proposition I, has more than one solution. Hence it is easy to see that it has both an odd solution and another twice that one.

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SHORTER NOTICES.

The Theory of Determinants; in the Historical Order of Development. Second edition, revised. By THOMAS MUIR. London, Macmillan and Co., 1906. xii + 491 pp.

THE present work "is intended for the student who aims at acquiring such a knowledge as can only be got by a study of the subject in the historical order of its development, for the investigator who is specially interested in this branch of mathematics and wishes to become acquainted with the various lines of attack opened up by previous workers, and for the general working mathematician who requires guide books and books of reference concerning special domains."* It covers the period from the beginning of determinants in 1693 to the close of 1841. Jacobi's long and exhaustive memoirs published in volumes 27 and 29 of *Crelle's Journal* are also considered in connection with skew determinants, although they appeared three or four years after the close of the period under consideration.

About one-third of the space is devoted to quotations from the works of the authors under consideration. These quotations enable the reader to form an independent judgment in reference to questions of priority and are especially valuable to those who do not have access to large libraries. Each new theorem of importance is numbered in Roman figures, and if the same result is either generalized or obtained in a different way by a subsequent writer it is marked among the contributions of the latter with the same Roman figure followed by an Arabic numeral.

Two very useful tables are inserted, showing the advance of the subject from 1693 to 1812 and from 1813 to 1841 respectively. There is also an index to the numbered results, but there is no general subject index. Its value as a reference book would have been greatly enhanced by such an index, especially since the chronological order of arrangement made it frequently impossible to bring together the material relating to the same topic.

The book is divided into two parts. Part I, in ten chapters, treats of general determinants and occupies considerably more than half the space. Complete chapters are devoted to the memoirs appearing in each of the years 1812 and 1841, while only one is given to those which appeared in the eighty-seven years from 1693 to 1779. Part II is devoted to special determinants which began to be studied, though without the present notation, by Lagrange and Gauss as early as 1773 and 1801 respectively. One chapter is devoted to each of the following subjects: axisymmetric determinants, alternants, jacobians, skew determinants, orthogonants.

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* From the preface.

Chapter XVI treats the miscellaneous special forms which prior to 1841 had received but little attention. The most fertile originator of such forms was Wronski, a poor Polish enthusiast whose egotism and wearisome style tended to attract few followers. The chapter is mainly devoted to the four special forms of determinants studied by Wronski, three of which were introduced by him. The other writers considered in this chapter are Scherk, Schweins, Jacobi, and Sylvester, the first two in relation to extensions of Wronski's results, the last two in relation to special forms of determinants arising from Bezout's method of eliminating the unknown between two equations of the nth degree and from Sylvester's dialytic process.

The last chapter consists of a single paragraph, devoted to a retrospect on special forms. Two other brief chapters (V and X) also are devoted to retrospects. In the former of these the fact is emphasized that nearly all of the earlier theorems are due to French writers, while the latter notes the rapid increase of interest after 1812 — especially in Germany, where Jacobi and Schweins took the most prominent part in the development of the subject. This spread of interest was largely due to the study of determinants whose elements have special forms. The beauty of some of these results made the notation more attractive and it spread rapidly into common use. Its great advantages as a thought saver, replacing the repeated thought tensions by a single demonstration, soon won for it universal favor, and, in some instances, undue prominence.

As the author of the present volume has published the most complete bibliography of determinants * and has contributed towards its advancement during a period of more than twenty years, we naturally expect the work to have the charm of ripe

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* *Quar. Jour. of Mathematics*, vols. 18, 21, 36. The bibliography includes 1,744 titles and covers the periods from the earliest writings on the subject down to the close of the nineteenth century.

scholarship. This expectation is enhanced by the fact that the author confines himself to an early period and that the present volume is a second edition of a valuable work that has been before the public for sixteen years. The typographical errors are few and insignificant. The most serious noticed occurs on page 4, where it is stated that the earlier edition appeared in 1900, instead of 1890.

In 1838 Liouville demonstrated a property of the special forms which were afterwards called wronskians by Muir (1881) and later writers. According to Anissimov * this seems to be the first important result relating to these useful forms, but the paper is not mentioned by our author although it falls within the period covered. Neither is it included in the bibliography cited above.

The present volume is almost twice as large as the first edition and includes a number of papers not there mentioned. The greatest difference between the two editions is in the facts that the papers relating to special forms are brought together in the new edition and that a table showing the advance of determinants from 1813 to 1841 has been added. The use of smaller type for quotations has greatly improved the appearance of the text. The completeness and attractiveness of the book combine to make it indispensable to the student of determinants and their history.
G. A. MILLER.

A First Course in Analytical Geometry, Plane and Solid, with Numerous Examples. By CHARLES N. SCHMALL. New York, Van Nostrand, 1905. 8vo. 7 + 318 pp.

ON the whole, this little book resembles so closely the ordinary text on the subject, both as to material and treatment, that a detailed account is scarcely necessary. Mention of the few points in which it does depart from the general type of text will be sufficient. In the chapter on loci, the usual examples from physics, economics, statistics, etc., are omitted, and attention is confined exclusively to purely geometric problems. This omission will probably be regarded as a fault or a merit according to one's views on the recently discussed relations of mathematics to physics and other subjects. The chapter on the circle precedes that on the transformation of coordinates, reversing the usual order. A short chapter on confocal conics is inserted, while the treatment of higher plane curves and of solid geometry are unusually brief.

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* Encyclopédie des Sciences Mathématiques, Tribune publique I, 1906, p. 3.

The most interesting novel feature is the early introduction of the determinant notation and its continued use throughout the book. In the first equation of the straight line (*i. e.*, in terms of the coordinates of two known points) the determinant form is given side by side with the explicit equation. Even if the student has had no previous treatment of the determinant, a few minutes of class-room explanation will enable him to grasp this simple form and the continued use will certainly give him that realizing sense of the geometric value of this notation which he too often fails to get in his formal course in algebra. E. B. COWLEY.

Lehrbuch der analytischen Geometrie. Erster Band: *Geometrie in den Grundgebilden erster Stufe und in der Ebene.* Von L. HEFFTER und C. KOEHLER. Leipzig, Teubner, 1905. 8vo. 16 + 526 pp.

THIS volume is in strong contrast to the book just noticed. While the authors aim to make the treatment elementary, they wish to give the student an introduction, at least, to the modern methods of relating the various parts and kinds of geometry into one comprehensive whole. They endeavor to follow the way suggested by Cayley's Sixth memoir on quantics and sketched by Klein in his Erlangen Programm. They consider that this can be done only by the early introduction of the transformation group, proceeding from the projective group to its subgroup, the affine, and then to its subgroup, the so-called "äquiform"; and not in the inverse order. This first volume contains the geometry in all spaces of one dimension, and in the plane. The second will be devoted to the finite "Bündel" and to ordinary space. In each case, the procedure is from projective to affine and then to æquiform geometry.

A condensed account of the contents will be useful in giving some idea of the ground covered. After an introductory chapter devoted to definitions and a few general considerations, there follows the first part, which is on geometry in spaces of one dimension. The first chapter relates to projective and affine geometry in the point range. The next takes up the quadratic equation and the point pair and its involution. This part is concluded by a consideration of the projective and æquiform geometries of the real pencils. The second section begins, under the heading of geometry in a plane, with a chapter on projective geometry in the plane and coördinates. The second chapter considers the principle of duality and projective theorems concerning points and straight lines. In the third, the projective and the reciprocal transformations of the plane are discussed. Then follow two chapters on the elements of the affine and æquiform geometries in the plane, and one on the general projective properties of curves of the second order and second class. After the projective classification of conics and the consideration of polarity in reference to conics, come three chapters treating of pencils and ranges of conics, the first in projective and the other two in affine geometry. The twelfth chapter takes up the æquiform geometry of conics. The next two deal with principal axes and foci and focal properties. In the last chapter is found the æquiform geometry of systems of conics (ranges and pencils). The book is concluded by an appendix on determinants. No knowledge of the calculus is presupposed.

The original plan was to begin with a systematic treatment of the axioms and base all subsequent work upon them. However, consideration for the mathematical immaturity of the students has led to some modifications; for example, the lengthy and abstract discussion of the axioms is omitted and a few theorems are inserted without proof. One naturally asks about the treatment of cross ratio. It is defined (page 36) in terms of the ratios of segments which have been measured in the ordinary way. With this metrical definition, it is used in projective geometry. In the footnotes, however, references are given to books in which the student will find a purely projective treatment.

The question that arises as one reads is that of the advisability of beginning with the spaces of one dimension. Will the beginner really grasp the ideas, are they concrete enough for his full comprehension, or will he fail to appreciate the purpose of the discussion, even though he may follow the separate details? In other words, it is a logical order; but, is it psychological? These are questions that can, perhaps, be answered only after one has attempted to use the book as a text. We are too ready to assume that a method of approach that is unusual will necessarily be difficult for the student. The style throughout is clear and simple, as well as stimulating and suggestive. The general plan seems admirable; and the student should have mastered in the end not only the usual collection of time-honored facts about conics, but a few of the well-known theorems such as Desargues's, Brianchon's, and Pascal's, as well as an introductory idea, at least, of that most important geometric concept, — the group of all projective transformations.

E. B. COWLEY.

Tratado de las Curvas Especiales Notables. By F. GOMES TEIXEIRA. Madrid, "Gaceta de Madrid," 1905, ix + 632 pp.

THIS volume had its inception in a prize problem proposed in 1892, and repeated in 1895, by the Royal Academy of Sciences of Madrid, requiring "An orderly list of all the curves of every kind to which definite names have been assigned, accompanying each with a succinct exposition of its form, equations and general properties, and with a statement of the books in which, or the authors by whom, it was first made known." This programme our author has closely adhered to except in one particular. To attempt to give the properties of all such curves would be extremely difficult and would make the resulting work unwieldy, he has therefore wisely limited himself to a list of over one hundred curves so selected as to include almost all of especial importance.

This treatise and Loria's work, "Spezielle algebraische und transcendente ebene Kurven," which appeared a little earlier, cover almost the same field. Both authors seem to have taken their suggestion from the theme of the Royal Academy. Teixeira, however, has followed that programme more closely. Loria's work is arranged in a more satisfactory manner and is somewhat more advanced in treatment. Teixeira's has the advantage of giving a considerable discussion of space curves.

The first two chapters (98 pages) of Teixeira's treatise are devoted to a detailed exposition of the properties of the most important cubic curves. In the third, fourth, and fifth chapters (158 pages) he treats of quartics and in the sixth chapter (68 pages) of algebraic curves of order higher than the fourth. He considers in the seventh chapter (39 pages) a number of transcendental curves, most of them of considerable physical importance. The spirals are considered in Chapter VIII (47 pages), the parabolas and hyperbolas $y = a^{1-k}x^k$ in Chapter IX (10 pages), and the cycloidal curves in Chapter X (56 pages). Chapter XI (45 pages) is devoted to the discussion of several plane curves of lesser importance. An exposition of the properties of a number of space curves is given in Chapters XII and XIII (72 pages). These chapters are not, however, up to the standard of the rest of the book. Chapter XIV (13 pages) is devoted to polhodes and herpolhodes.

The treatment is, throughout, quite elementary, and can be followed by anyone with a knowledge of analytics and calculus. The book is not, however, suitable for a student seeking a systematic treatment of the theory of the higher curves, as its aim is the consideration of noteworthy curves and not the systematic exposition of curves in general. It is a treatise, not on curve theory, but on particular curves.

The method of exposition, whenever practicable, is as follows: the rectangular and polar equations of the curve are given, the form of the curve is derived from the equations and a figure of the curve is shown. The most interesting geometric properties of the curves are then deduced, the parametric equations — when the curve is unicursal — derived and the integrals for the length of arc and the area of the curve obtained.

The book is metrical both in viewpoint and in method. Trilinear coordinates are, however, used a few times and line coordinates several times but not in such a manner as to necessitate a previous knowledge of those subjects. It is to be regretted that the author does not enter into the projective theory at least enough to show the projective interrelations, and occasionally the projective identity, of some of the curves considered.

The content and method of the book are such as to make it especially valuable for engineers and others not specialists in geometry, but geometers also will find in it a valuable collection of information on particular curves. The Spanish text presents little difficulty to one who can read any Romance language with facility. C. H. SISAM.

The Scientific Papers of J. WILLARD GIBBS. Volume 1, *Thermodynamics*, xxviii + 434 pp., with portrait; volume 2, *Dynamics, vector analysis and multiple algebra, electromagnetic theory of light, etc.*, ix + 284 pp. London, Longmans, Green, and Co., 1906. Large 8vo.

SHORTLY before the death of Professor J. Willard Gibbs he had decided to yield to requests from various sources and to reprint his principal papers on thermodynamics with some additional chapters, upon which he was then at work. After his sudden death had cut short his plans, his family resolved to fulfil his intention, as far as might be, by printing, under the editorial direction of Professor Bumstead and Dr. Van Name, a complete edition of his published papers * with such additions as could be made out from his brief and meager notes.

The first volume of the Scientific Papers contains the collection of articles on thermodynamics with seventeen supplementary pages of unpublished fragments which represent a first draft of a part of the material that Professor Gibbs was preparing. Among the subjects which he was intending to treat were:

I. On the values of potentials in liquids for small components.
II. On the fundamental equations of molecules with latent differences.
III. On the fundamental equations for vanishing components.
IV. On the equations of electric motion.
V. On the liquid state, $p = 0$.
VI. On entropy as mixed-up-ness.
VII. Geometric illustrations.
VIII. On similarity in thermodynamics.
IX. Cryohydrates.

Of these, his notes contained but the partial treatment of I and IV, although VI may be regarded as somewhat discussed in chapters XII to XV of the Statistical Mechanics. One term which attracts the attention of the reader as probably something of great importance is *latent differences.* Unfortunately the notes left by Professor Gibbs do not permit of any definite conclusions as to what he meant. This must be a cause of deep regret to the scientific world — a feeling which may be somewhat ameliorated by the fact that at last the complete works of Willard Gibbs are easily available for reading and reference.

The second volume contains all the other papers, among which may especially be noted the original pamphlet on Vector Analysis, privately printed in 1881–4; an unpublished letter on the use of vector methods in the determination of orbits, written to Hugo Buchholz; and a short abstract, bearing the date 1884, of a communication to the American association for the advancement of science on the subject of statistical mechanics, which shows that even at that early date the author was tolerably far advanced on the investigations which he published only in 1902. It is regrettable that some account of the later researches on multiple algebra were not available for publication, in particular the matter concerning double products which constitutes an important advance in the general theory. Professor Gibbs seems to have kept this, like many other ideas certainly in an advanced state of development, in his mind rather than on paper. For this reason his published work which has now appeared contains practically all that the world can ever hope to know of his investigations.

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* With the exception of his two books in the Yale Bicentennial Series: Vector Analysis (1901), edited by E. B. Wilson, and Statistical Mechanics (1902), Charles Scribner's Sons, New York City.

E. B. WILSON.

Leibnizens Nachgelassene Schriften physikalischen, mechanischen und technischen Inhalts. Herausgegeben von DR. ERNST GERLAND. Leipzig, Teubner, 1906, vi + 256 pp.

THE volume contains hitherto unpublished papers of Leibniz on physics and mechanics, mostly written during the interval 1670 to 1686, or between the 24th and 40th year of his life. They are partly in Latin, partly in German and partly in French. The editor has supplied historical and explanatory notes. These articles are of interest, not only as showing that Leibniz sometimes touched upon questions which have since become of fundamental importance, such as the conservation of energy, but also as showing the progress of his ideas and his method of work.

FLORIAN CAJORI.

NOTES.

THE *American Mathematical Monthly* will hereafter be edited by Professors B. F. FINKEL and H. E. SLAUGHT, with the cooperation of Professor L. E. DICKSON. Since August, 1906, the *Monthly* has been published under the auspices of the University of Chicago.

AT the meeting of the London mathematical society held on December 13, the following papers were read: By C. S. JACKSON, "On the form of a surface of a searchlight reflector"; by L. F. RICHARDSON, "The potential equation and others with function given on the boundary"; by J. MERCER, "On the limits of real variants"; by E. W. BARNES, "The asymptotic expansion of integral functions defined by generalized hypergeometric series"; by P. A. MCMAHON, "The diophantine equation $x^n - Ny^n = z$.

AT the meeting of delegates of associations of teachers of mathematics and the natural sciences held at Columbia University on December 27, it was decided to organize an American federation of such associations. A committee, with Professor T. S. FISKE as chairman and Professor C. R. MANN as secretary, was appointed to take charge of the preliminary steps.

A FIFTH installment of the second part of volume 10 of the *Jahresbericht der Deutschen Mathematiker-Vereinigung*, containing pages 1073–1392 of Burkhardt's Bericht über die Theorie der oscillierenden Funktionen has just appeared from the press of Teubner, in Leipzig. This does not complete the volume, as promised, however, but treats of English advances in mathematical physics up to the time of Kelvin, and the subsequent development of the theory of heat. It is proposed in the next installment to treat of each recent branch separately.

THE firm of B. G. Teubner in Leipzig announces that the following volumes of the *Mathematische Annalen*, the first edition of which was exhausted and long out of print, have been reprinted and can now be obtained for 28 marks each: volumes 3, 4, 5, 6, 7, 32, 33, 50, 51, 52, 53.

A BUST of the late Professor GUIDO HAUCK has been unveiled in the court of the main building of the technical school at Charlottenburg.

A TABLET has been placed in Memorial Hall of Yale University in memory of ELIAS LOOMIS, LL.D., professor of natural philosophy and astronomy from 1860 to 1889.

AT the Collège de France Dr. P. BOUTROUX has been appointed to the special lectureship established a few years ago by M. Peccot under the condition that the appointee should be under thirty years of age. The tenure of office is three years. The preceding holders of the office are Professors Baire, Borel, and Lebesgue and Dr. Servant.

PROFESSOR A. E. H. LOVE, of Oxford University, has been elected president of Section A (mathematics and physics) for the Leicester meeting of the British Association for the advancement of science.

PROFESSOR L. BIANCHI, of the University of Pisa, has been elected a member of the Royal institute of Venice.

PROFESSOR W. FIEDLER, of the technical school at Zürich, has been elected corresponding member of the academy of sciences at Munich.

MR. T. FRIESENDORFF, docent in the University at St. Petersburg, has been appointed associate professor of applied mathematics at the technical school at St. Petersburg.

PROFESSOR M. LERCH, of the University of Freiburg, Switzerland, has been appointed professor of mathematics at the Bohemian technical school at Brünn.

PROFESSOR V. OPPOLZER, of the University of Innsbruck, has been promoted to a full professorship of astronomy at the same institution.

PROFESSOR F. STREINTZ has been promoted to a full professorship of mathematics at the technical school at Graz.

PROFESSOR H. DE VRIES, of the technical school at Delft, has been appointed professor of mathematics at the University of Amsterdam.

Dr. N. Nielsen, of the University of Kopenhagen, has been promoted to an associate professorship of mathematics, and elected foreign member of the academy of sciences of Halle.

Dr. H. de S. Pittard has been appointed assistant lecturer in mathematics at King's College, London.

Mr. G. G. Chambers has been appointed instructor in mathematics at the University of Pennsylvania.

Dr. A. W. Panton, lecturer in mathematics at Trinity College, Dublin, and joint author of Burnside and Panton's Theory of Equations, died December 18, at the age of 60 years.

Dr. Thomas Harrison, professor of mathematics and later chancellor of the University of New Brunswick, died September 18, 1906, at the age of 68 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Bendt (F.). Differential- und Integralrechnung. 3te Auflage. Leipzig, 1906. 8vo. Cloth. M. 3.00

Bürklen (O. T.). Aufgabensammlung zur analytischen Geometrie des Raumes. Leipzig, Goschen, 1906. 16mo. 98 pp. Cloth. (Sammlung Göschen.) M. 0.80

Burkhardt (H.). Funktionentheoretische Vorlesungen. 2te Auflage. Vol. II: Elliptische Funktionen. Leipzig, Veit, 1906. 8vo. 16 + 373 pp. M. 10.00

Campbell (D. F.). A short course on differential equations. New York, Macmillan, 1906. 12mo. 8 + 96 pp. Cloth. $0.90

Eggenberger (J.). Beitrage zur Darstellung des Bernoullischen Theorems der Gammafunktion und des Laplaceschen Integrals. 2te Auflage. Jena, Fischer, 1906. 8vo. 79 pp. M. 2.50

Fields (J. C.). Theory of the algebraic functions of a complex variable. Berlin, Mayer & Muller, 1906. 8vo. 7 + 186 pp. M. 12.00

Fuchs (L.). Gesammelte mathematische Werke. Herausgegeben von R. Fuchs und L. Schlesinger. Vol. II: Abhandlungen 1875–1887, redigiert von L. Schlesinger. Berlin, Mayer & Müller, 1906. 8vo. 10 + 487 pp. M. 30.00

Grünbaum (H.). Lehr- und Uebungsbuch der Differenzialrechnung für mittlere technische Lehranstalten, Realgymnasien, Oberrealschulen u. s. w., sowie zum Selbststudium. 2te, verbesserte Auflage. Würzburg, Frank, 1907. 8vo. 8 + 119 pp. M. 1.75

Kozák (J.). Grundprobleme der Ausgleichungsrechnung nach der Methode der kleinsten Quadrate. Vol. I. Wien, 1906. 8vo. M. 11.00

Laurent (H.). La géométrie analytique générale. Paris, Hermann, 1906. 8vo. 7 + 153 pp.

Nielsen (N.). Theorie des Integrallogarithmus und verwandter Transzendenten. Leipzig, Teubner, 1906. 8vo. 6 + 106 pp. M. 3.60

Pincherle (S.). Algebra complementare. Parte I: Analisi algebrica. 2a edizione. Milano, Hoepli, 1907. 16mo. 8 + 174 pp. (Manuali Hoepli.) L. 2.00

Re (A. del). Lezioni sulle forme fondamentali dello spazio rigato, sulla dottrina degli immaginari e sui metodi di rappresentazione nella geometria descrittiva. Napoli, Alvano, 1906. 8vo. 4 + 85 pp. L. 3.50

Reisky. Zur Einführung in die geometrische Analysis. Leobschutz, 1906. 4to. 11 pp. M. 1.50

Reye (T.). Die Geometrie der Lage. Vorträge. Abteilung II. 4te, umgearbeitete und vermehrte Auflage. Stuttgart, Kröner, 1907. 8vo. 8 + 335 pp. M. 10.00

Schlesinger (L.). See Fuchs (L.).

Schröder (R.). Aufgaben der Quadratur der Kegelschnitte. Gross-Lichterfelde, 1906. 8vo. 19 pp. M. 1 50

Vegas (M.). Tratado de geometría analítica. Vol. I. 2a edición. Madrid, Fortanet, 1906. 8vo. 6 + 688 pp. P. 15.00

II. ELEMENTARY MATHEMATICS.

August (E. F.). Vollständige logarithmische und trigonometrische Tafeln. 28ste Auflage. Leipzig, 1906. 8vo. 8 + 204 pp. Cloth. M. 1.60

Borel (E.). Arithmétique (premier cycle) Paris, Colin, 1907. 16mo 7 + 221 pp. (Cours de mathématiques rédigé conformément aux nouveaux programmes.) Fr 2.50

Bork (H.). Mathematische Hauptsätze Ausgabe für Realgymnasien und Oberrealschulen. Nach dem Tode des Verfassers herausgegeben von M. Nath. Teil I: Pensum der Unterstufe (bis zur Untersekunda einschliesslich). 5te, durchgesehene und den preussischen Lehrplänen von 1901 angepasste Auflage Leipzig, Dürr, 1907. 8vo. 252 pp. Boards. M. 2.50.

Budinich (A.). See Heis (E.).

Büttner (A.). Die Elemente der Buchstabenrechnung und Algebra. Nebst einem Anhange, enthaltend Logarithmentafeln für die Zahlen von 1 bis 10,000. Für den Schul- und Selbstunterricht. 18te Auflage. Neubearbeitet von W. Petri. Bielefeld, Velhagen & Klasing, 1907. 8vo. 8 + 287 pp. Boards. M. 4.00

DAVIDSON (J.). Arithmetic and algebra. London, Hodder, 1906. 12mo. (Self-educator series.) 1s.

DURELL (C. V.). Elementary problem papers. London, Arnold, 1906. 12mo. 128 pp. Cloth. 1s. 6d.

FAILOR (I. N.). Plane and solid geometry. New York, Century Company, 1906. 12mo. 12 + 418 pp. Cloth. $1.25

FEIST (A.). Mathematische Aufgaben. Verzeichnis der bei den Reifeprüfungen der neunstufigen höheren Lehranstalten des Herzogtums Braunschweig 1892–1903 bearbeiteten Aufgaben. Abteilung II. Braunschweig, 1906. 4to. 15 pp. M. 1.00

FLECHSIG. See LUCZAK.

FRENCH (C. H.) and OSBORN (G.). Graphs, or the graphical representation of algebraic functions. 2d edition, revised and enlarged. London, Clive, 1906. 12mo. 136 pp. Cloth. 1s. 6d.

FUCINI (C.). Geometria piana e nozioni di geometria solida per le scuole secondarie inferiori. 8a edizione. Genova, Gioventù, 1907. 8vo. 143 pp. L. 1.50

FUSS (K.). Die wichtigsten Satze der Planimetrie und Stereometrie. Für den Schul- und Selbstunterricht. 2te, stark vermehrte und vollständig umgearbeitete Auflage. Nürnberg, Korn, 1907. 8vo. 5 + 142 pp. M. 2.00

HARTL (H.). Lehrbuch der ebenen Trigonometrie. Für den Unterrichtsgebrauch und für das Selbststudium. Mit uber 500 Uebungsbeispielen nebst deren Resultaten. 2te, umgearbeitete Auflage. Wien, Hölder, 1906. 8vo. 3 + 106 pp. Boards. M. 1.44

HECHT (C.). Lehrbuch der elementaren Mathematik. Als Ergänzung zu "Rechenbuch für Madchenschulen" bearbeitet. Teil II: Arithmetik. Bielefeld, Velhagen & Klasing, 1906. 8vo. 6 + 156 pp. Boards. M. 1.80

HEIS (E.). Raccolta di esempi e quesiti di aritmetica ed algebra ordinati ad uso delle scuole secondarie. Versione di A. Budinich. 3a edizione. Torino, Loescher, 1907. 8vo. 4 + 387 pp. L. 4.00

JAAKSON (J.). Graphische Multiplikations-, Divisions-, Quadraten- und Wurzeltabelle. Blatt I: Für 1-, 2- und 3-stellige Zahlen. Berlin, 1906. Sq. 16mo. 2 pp. M. 0.50

LANNER (A.). Neuere Darstellungen der Grundprobleme der reinen Mathematik im Bereiche der Mittelschule. Berlin, Salle, 1907. 8vo. 8 + 192 pp. M. 3.00

LONDON matriculation mathematics papers. From January, 1896, to September, 1906. London, Clive, 1906. 12mo. 130 pp. Cloth. 1s. 6d.

LUCHERINI (R.). Elementi di aritmetica e geometria. Parte I, per gli alunni della IV classe elementare. Firenze, Pacetti, 1906. 16mo. 144 pp. L. 0.60

——. Parte II, per gli alunni della V classe elementare. Firenze, Pacetti, 1906. 16mo. 182 pp. L. 0.80

Luczak und Flechsig. Die Dreiteilung des Winkels. Neue, mathematisch genaue Losung mit Hilfe des Luczakzirkels. Lissa, 1906. 8vo. M. 0.30

Marks (C. I.). Mathematical questions and solutions from the *Educational Times.* New series, Vol. 10. London, Hodgson, 1906. 8vo. Cloth. 6s. 6d.

Meray (C.). Nouveaux éléments de géométrie. 3e édition (réduction et refonte partielle de l'édition de 1903). A l'usage des écoles primaires supérieures, des écoles normales d'instituteurs, des écoles pratiques de commerce et d'industrie, des écoles professionnelles et de toutes les classes élémentaires dans les établissements d'enseignement secondaire. Dijon, Jobard, 1906. 8vo. 20 + 309 pp.

Nath (M.). See Bork (H.).

Osborn (G.). See French (C. H.).

Palmieri (F. S.). Elementi di aritmetica e di algebra per le scuole secondarie superiori. Parte I. Messina, Secolo, 1906. 16mo. 6 + 240 pp. L. 3.00

Pasquali (P.). Geometria intuitiva, ad uso delle scuole elementari superiori, tecniche, normali e industriali. Lezioni di ritaglio geometrico. Parte III. Milano, Vallardi, 1906. 16mo. 39 pp. L. 0.50

Petri (W.). See Büttner (A.).

Riechemeyer. Das erste Jahr des planimetrischen Unterrichtes am Gymnasium. Gütersloh, 1906. 8vo. 20 pp. M. 1.50

Roeder (H.). Trigonometrische und stereometrische Lehraufgabe der Unter-Sekunda (Prima der Realschulen). Sonderabdruck aus der Kamblyschen Planimetrie. 4te, vermehrte Auflage. Breslau, Hirt, 1906. 8vo. 4 + 49 pp. M. 0.60

Ruiz Tapiador (A.). Nociones y ejercicios de aritmética y geometría. Toledo, Serrano, 1906. 8vo. 108 pp.

Salomon (A.). Leçons de géométrie, à l'usage de l'enseignement secondaire des jeunes filles. 3e édition. Paris, Vuibert et Nony, 1906. 16mo. 230 pp.

Schächer (W.). Das Wichtigste aus der geometrischen Formenlehre. 2te, verbesserte Auflage. Sternberg, Hitschfeld, 1906. 8vo. 39 pp. M. 0.36

Schmid (A.). See Sickenberger (A.).

Sibirani (F.). Elementi di algebra per le scuole tecniche e normali, riveduti dal prof. Cesare Arzelà. Firenze, Le Monnier, 1906. 16mo. 151 pp. L. 1.50

Sickenberger (A.). Uebungsbuch zur Algebra. Abteilung I: Erste und zweite Stufe der Rechnungsarten einschliesslich der linearen Gleichungen mit einer und mehreren Unbekannten. 5te Auflage, bearbeitet von A. Schmid. Munchen, Ackermann, 1906. 8vo. 5 + 106 pp. Boards. M. 1.35

Trucco (E.). Elementi di aritmetica ragionata, ad uso del ginnasio superiore. Treviglio, Messaggi, 1906. 8vo. 122 pp. L. 1.60

Tuñon de Lara (M.). Lecciones de geometría. 1a edición. Toledo, Serrano, 1906. 8vo. 282 pp. P. 10.00

Winter (W.). Stereometrie. Lehrbuch und Aufgabensammlung für Schulen. 4te Auflage. München, Ackermann, 1906. 8vo. 4 + 116 pp. M. 1.60

III. APPLIED MATHEMATICS.

Ahrens (C.). See Lauenstein (R.).

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THE THIRTEENTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE thirteenth annual meeting of the Society was held in New York City on Friday and Saturday, December 28–29, 1906, forming a part of the general gathering of scientists in attendance at the meetings of the American Association for the Advancement of Science and the numerous affiliated societies. A noticeable effect of this reinforcement of interest was an enlarged and more widely representative attendance, considerably exceeding that of any previous meeting of the Society. Friday morning was devoted to a joint session with Section A of the Association and the Astronomical and Astrophysical Society. Professor Simon Newcomb presided; seven papers selected from the programmes of the participating organizations were presented before a large audience. The remaining three sessions were taken up with the regular programme, to which were also added several papers from Section A. Unfortunately the limited time did not permit of adequate presentation and discussion of even the most important papers announced, several being finally read by title. The productive capacity of the Society is constantly outrunning the facilities for presentation; a stricter enforcement of the rules regarding the early submission of titles and abstracts of papers, and a rigorous observance of the order of presentation announced on the printed programme are demanded to relieve the congestion and preserve the orderly conduct of the meetings. It has also been strongly urged that to promote intelligent discussion of the papers abstracts should be included in the printed programmes. This would certainly be a desirable feature. Its practical working out would require some increase and redistribution of administrative forces and responsibilities. The whole subject will receive the serious consideration of the Council.

The attendance at the several sessions of the annual meeting included the following eighty-one members of the Society:

Mr. William Betz, Mr. W. C. Brenke, Professor E. W. Brown, Professor W. G. Bullard, Dr. W. H. Bussey, Dr. W. B. Carver, Dr. J. E. Clarke, Professor F. N. Cole, Miss E. B.

Cowley, Dr. H. N. Davis, Dr. W. S. Dennett, Professor T. W. Edmondson, Professor F. C. Ferry, Professor J. C. Fields, Professor W. B. Fite, Professor A. S. Gale, Dr. F. L. Griffin, Miss Ida Griffiths, Professor G. B. Halsted, Professor Harris Hancock, Professor J. G. Hardy, Professor H. E. Hawkes, Professor E. R. Hedrick, Professor C. S. Howe, Professor L. S. Hulburt, Professor E. V. Huntington, Professor J. I. Hutchinson, Professor Harold Jacoby, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Professor W. W. Landis, Dr. G. H. Ling, Mr. L. L. Locke, Dr. W. R. Longley, Professor T. E. McKinney, Professor J. McMahon, Professor James Maclay, Dr. Isabel Maddison, Professor H. P. Manning, Professor Max Mason, Professor Mansfield Merriman, Professor W. H. Metzler, Professor H. B. Mitchell, Dr. C. L. E. Moore, Dr. R. L. Moore, Professor Frank Morley, Professor Richard Morris, Professor Simon Newcomb, Professor G. D. Olds, Professor James Pierpont, Professor R. W. Prentiss, Miss Amy Rayson, Mr. H. W. Reddick, Miss S. F. Richardson, Professor E. D. Roe, Miss Ida M. Schottenfels, Professor Arthur Schultze, Mr. E. I. Shepard, Mr. L. P. Siceloff, Professor D. E. Smith, Professor P. F. Smith, Professor Virgil Snyder, Professor H. F. Stecker, Dr. R. P. Stephens, Professor Ormond Stone, Professor W. E. Story, Dr. W. M. Strong, Professor H. D. Thompson, Mr. M. O. Tripp, Professor H. W. Tyler, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Professor Oswald Veblen, Professor L. A. Wait, Mr. H. E. Webb, Professor A. G. Webster, Professor L. G. Weld, Professor H. S. White, Professor E. B. Wilson, President R. S. Woodward.

Professor E. B. Van Vleck presided at the afternoon session on Friday, being relieved by Professor Morley. President H. S. White, Professor Morley, and Vice-President P. F. Smith occupied the chair at the Saturday sessions. The Council announced the election of Mr. E. I. Shepard, of Harvard University, to membership in the Society. Eight applications for membership were received.

Owing to recent illness, President Osgood was unable to deliver his address; it will be presented at the April meeting of the Society.

The organization of a new Section, to be known as the Southwestern Section of the Society, was authorized by the Council. It was decided to hold the next summer meeting at Cornell

University. The next annual meeting was fixed for Friday and Saturday, December 27–28, 1907. An amendment to the constitution was adopted by which the Editorial Committee of the *Transactions* are included in the membership of the Council.

Reports were submitted by the Treasurer, Librarian, and Auditing Committee. These reports will appear in the forthcoming Annual Register. The membership of the Society has increased during the year from 504 to 548, including at present 50 life members. The number of papers presented at all meetings during the year, including the preliminary meeting of the Southwestern Section was 176, as against 147 in 1905. The total attendance of members was 350; 192 members attended at least one meeting during the year. The library now contains about 2,500 bound volumes; a list of the journals and of the accessions during 1905–1906 is printed in the Annual Register. The Treasurer's report shows a balance of $5,195.80 on hand December 14, 1906; of this balance $2750.06 is credited to the life-membership fund. Sales of the Society's publications during the year amounted to over $1,500.

The publication of the lectures delivered by Professors Moore, Mason, and Wilczynski at the New Haven Colloquium, September, 1906, has been generously undertaken by Yale University.

At the annual election, which closed on Saturday morning, the following officers and other members of the Council were chosen:

President,	Professor H. S. White.
Vice-Presidents,	Professor Heinrich Maschke, Professor P. F. Smith.
Secretary,	Professor F. N. Cole.
Treasurer,	Dr. W. S. Dennett.
Librarian,	Professor D. E. Smith.

Committee of Publication,

Professor F. N. Cole,
Professor Alexander Ziwet,
Professor D. E. Smith.

Members of the Council to serve until December, 1909,

Professor G. A. Bliss,
Professor E. W. Brown,
Professor M. W. Haskell,
Professor A. G. Webster.

The informal dinner, always arranged in connection with each meeting of the Society, adds much to the pleasure of these occasions. Despite other distractions connected with the general gathering of scientists, over forty members attended the dinner on Friday evening and passed a few pleasant hours in social intercourse and renewal of old acquaintance.

The following papers were read at this meeting:

(1) Professor S. E. Slocum: "The rational basis of mathematical pedagogy."

(2) Dr. F. L. Griffin: "On the law of gravitation in the binary systems."

(3) Professor James McMahon: "A differential property of the lamellar vector field."

(4) Professor J. I. Hutchinson: "A method of constructing the fundamental region of a discontinuous group of linear transformations."

(5) Professor James Pierpont: "Multiple integrals" (preliminary communication).

(6) Professor Oswald Veblen: "Collineations in a finite projective geometry."

(7) Dr. W. R. Longley: "Some particular solutions in the problem of n bodies."

(8) Professor Max Mason: "The expansion of an arbitrary function in terms of normal functions."

(9) Professor R. D. Carmichael: "On Euler's ϕ-function."

(10) Dr. Arthur Ranum: "On the group of classes of congruent matrices."

(11) Dr. W. B. Carver: "Sets of quadric spreads connected with the configuration $\Gamma^{\nu}_{n,r}$."

(12) Professor C. J. Keyser: "Circle range transversals of circle ranges in a plane: a problem of construction."

(13) Professor C. J. Keyser: "Concerning the analytic treatment of geometric involution."

(14) Dr. A. B. Coble: "A generalization of the plane Hesse configuration."

(15) Dr. A. B. Coble: "Involutory Cremona transformations."

(16) Professor W. E. Story: "Denumerants of double differentiants."

(17) Professor Virgil Snyder: "Birational transformations of curves of high genus."

(18) Professor T. E. McKinney: "On the continued fractions representing properly and improperly equivalent real numbers in a system of continued fractions depending on a variable parameter."

(19) Professor H. E. Hawkes: "On elementary divisors."

(20) Professor E. B. Wilson: "Rotations in higher dimensions."

(21) Professor Edward Kasner: "Systems of extremals in the calculus of variations."

(22) Professor Edward Kasner: "The motion of a particle in a resisting medium."

(23) Dr. R. P. Stephens: "Note on a system of curves of class n and order $2(n-1)$."

(24) Dr. D. C. Gillespie: "On the construction of an integral of Lagrange's equation in the calculus of variations."

(25) Mr. F. R. Sharpe: "The general circulation of the atmosphere."

The following papers from Section A were also placed on the programme of the Society:

Professor G. B. Halsted: "The sect carrier and the set sect."

Professor Harris Hancock: "On a fundamental theorem of Weierstrass by means of which the theory of elliptic functions may be established."

Professor G. A. Miller: "On the minimum number of operations whose orders exceed two in any finite group."

Dr. Gillespie's paper was communicated to the Society through Professor Snyder. Professor Slocum's paper was read by Professor Hancock. The papers of Professor Carmichael, Dr. Ranum, Dr. Coble, Professor Snyder, Professor Hawkes, Professor Wilson, Dr. Stephens, Dr. Gillespie, and Mr. Sharpe were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Professor Slocum's article points out the lack of any general principle of mathematical pedagogy, and the desirability of establishing such a principle. The fundamental principles underlying general pedagogy are first discussed and shown to rest upon the natural law which Herbert Spencer expressed by saying that the genesis of knowledge in the individual follows the same course as the genesis of knowledge in the race. This

leads directly to the historical method of presentation which forms the basis of all scientific modern pedagogy. In the case of mathematics this method applies with peculiar force, being in fact the logical as well as the psychological method of development.

As a suggestive instance of the application of the historical method, Herbart's culture epoch theory is summarized and its merits and defects pointed out. In applying the historical method to mathematics, it is shown that the difficulties encountered in elementary branches of the subject, such as geometry and algebra, arise from a lack of harmony between the culture from which these subjects originated and the spirit of modern civilization. In this connection the culture of the Greeks, and also of the Hindoos, is characterized and contrasted with our own, thus making the nature of the difficulties encountered in these particular subjects clearly apparent. The length of time taken for the historical development of a subject is also shown to be an index of its difficulty, thus affording a means of making a quantitative, as well as a qualitative, estimate of the relative difficulty of various subjects. The various practical attempts to improve mathematical instruction, such as Grube's method of number analysis, the spiral method of instruction, the correlation of courses in the Prussian schools, the Perry movement, etc., are also studied, and shown to be but particular applications of the historical method, thus affording the best inductive proof of its validity. In conclusion, suggestions are made for a detailed application of the method to the elementary mathematical curriculum, based on the theory that the ideal arrangement of courses and subject matter depends simply on obtaining the proper historical perspective.

2. In studying the gravitational law in the binary systems, Dr. Griffin discards the more severe hypotheses hitherto made (*e. g.*, that the force is such as to make the orbit a conic for all initial conditions), and inquires what laws of force satisfy the observational data and the general hypotheses usually admitted. He shows that the only law of force under which the given orbits can be described, and satisfying the conditions: (*a*) the force is a single-valued function of the distance; (*b*) the force permits real orbits in all parts of the plane, is a type including the newtonian as a special case. Also, if the further condition be imposed, that within some circle the force be nowhere an in-

creasing function of the distance, the law of Newton is required.

A study of the trajectories for the more general type shows: that, for all initial velocities below a certain limit, the orbit is stable; and that through every point of the plane there passes in every direction one and but one orbit which is a conic. Various properties of the family of conics are pointed out; and certain exceptions, arising when the bodies are treated as having size, are discussed.

3. Professor McMahon's paper is in abstract as follows: Any lamellar field has the differential property expressed by the equation $d(\log q)/dn = d\theta/ds$, in which q is the length of the vector at a point P of the field, and $d\theta$ is the differential angle through which the vector turns as P moves through a distance ds along the curve of the vector, the other differential dn being taken along the principal normal to the curve just mentioned. A similar theorem holds in any two-dimensional lamellar field; and, in the particular case of a two-dimensional laplacian field, the differentials dn and ds may be taken in *any* two rectangular directions in the plane; this is in accordance with a well known property of two conjugate functions ($\log q$ and θ) in the complex plane. If a laplacian three-dimensional field is symmetric about an axis, it has the same differential property as a laplacian two-dimensional field if the differentiations are performed in a meridian plane.

4. The paper by Professor Hutchinson gives a simple and effective method for constructing the fundamental regions for a finite or infinite discontinuous group of linear transformations on n variables which leaves any given hermitian form unaltered.

5. Professor Pierpont's preliminary communication dealt with the relation of an improper multiple integral to its iterated integrals, in continuation of the author's paper in the *Transactions*, January, 1906.

6. Professor Veblen showed that the most general collineation of $PG(k, p^n)$ is of the form $x'_i = \sum_j a_{ij} x_j^{p^m}$, where $i, j = 1, 2, \cdots, k + 1$; $m = 0, 1, \cdots, n - 1$.

7. Dr. Longley considered some special cases of the plane motion of a given number of bodies. It is supposed that the

bodies form an invariable configuration which turns about the center of mass of the system. In the problem of four bodies it was shown that the rhombus is a possible configuration provided (1) opposite masses are equal, (2) the ratio of the diagonals of the rhombus is equal to or greater than unity and less than $\sqrt{3}$. The ratio of the masses is uniquely determined by the selection of the ratio of the diagonals. If a fifth body be introduced at the center of the figure, the same limits upon the ratio of the diagonals hold. A symmetric configuration was defined as one in which the masses are distributed symmetrically with respect to every line joining a body to the center of mass. A few special cases of symmetric configurations were examined. All cases in which the bodies may move in circles can be extended to motion in any conic, the configuration being always of the same shape, but of varying size.

8. In a previous paper Professor Mason gave a proof of the expansion of an arbitrary function in terms of normal functions of a certain differential equation of the second order. In that paper the convergence of the series was established by a method analogous to one used by Schmidt in connection with integral equations. In the present paper this is replaced by a simpler and more direct method.

9. Professor Carmichael's paper appeared in the February number of the BULLETIN.

10. Dr. Ranum's paper is a development of the theory begun in his article on the same subject, to appear in the *Transactions*. Among the theorems proved are the following: (*a*) Every chief binary group of classes of congruent matrices, mod $(p^{a_{ij}})(i, j = 1, 2; p = \text{prime})$, is simply isomorphic (in a certain particular way) with one and only one (normal) group, satisfying the conditions $a_{11} \geqq a_{22}$, $a_{22} \geqq a_{21} \geqq 0$, $a_{11} + a_{22} \geqq a_{12} \geqq a_{11}$. (*b*) The great majority of groups of classes of congruent matrices cannot be directly written as linear substitution groups, but every group for which the moduli of any two columns are proportional can be so written.

11. In an earlier paper,* in which the method of treatment was synthetic, Dr. Carver called attention to certain peculiarly related

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* "On the Cayley-Veronese class of configurations." *Transactions Amer. Math. Society*, vol. 6, No. 4 (October, 1905).

sets of conics connected with the plane configuration $\Gamma_{n,2}^{\nu}$. In the present paper this idea is treated analytically and is extended to the consideration of sets of quadric spreads connected with the r-dimensional configuration $\Gamma_{n,r}^{\nu}$. It is shown that, the quadric spreads being given, the configuration is determined. Special cases for $r = 1$ and $r = 3$ are then discussed.

12. Professor Keyser's first paper deals with an important detail in the plane geometry that arises from employing the circle range as primary element. A circle range is the ensemble of circles having two points in common. Two circle ranges having a common circle are transversals of each other. Four arbitrary circle ranges r_1, r_2, r_3, r_4 have in common two and but two transversal ranges, ρ_1 and ρ_2. These may be real and distinct or coincident or conjugate imaginary. The problem solved is that of constructing ρ_1 and ρ_2 in the case where they are real. The four ranges taken in sets of three determine four semiquadrics of ranges, viz., Q_{123}, Q_{124}, Q_{134}, Q_{234}. The transversals of the ranges of a Q constitute a complementary semiquadric Q' of that Q. The four Q's have ρ_1 and ρ_2 in common. The four Q's have no common range, but any three of them have one, any two of them two, common ranges. Any Q and its Q' together constitute a hyperboloidal configuration H of ranges, having thus two sets of generating ranges, those of Q and those of Q'. The four H's have ρ_1 and ρ_2, but no other ranges, in common. The radical axes of the ranges of a Q envelop a conic section. The like is true of a Q'. If a Q and a Q' belong to the same H, the two conics coincide. The four conics thus associated with the four H's have the radical axes of ρ_1 and ρ_2 in common. The conics are the intersections of the H's and the special congruence composed of the lines (infinite circles) of the plane. The conic as envelope is thus a second order locus of infinite circles. In like manner the point conic is the envelope of circle ranges of infinite circles. The four conics may be constructed by using the radical axes of ρ_1 and ρ_2 and those of the given r's taken three at a time. The problem of constructing ρ_1 and ρ_2 is *logically* the same as that of constructing the transversals of four lines of space, but the construction makes no use in euclidean fashion of any circle configuration analogous to the plane, a fact supporting the hypothesis that the plane need not be employed as element in the three-dimensional constructions.

13. The second paper by Professor Keyser deals with the question of a suitable algebra for geometric involution. Connecting the exposition (for the sake of convenience) with a range of points, the problem is to provide an analytic machinery that shall handle the point pair as element instead of the points of the pair. This is done in a manner analogous to that of Darboux and Klein in the matter of pentaspherical point and sphere coordinates in space. Three point pairs in mutual involution are taken for configuration of reference. The power of a point as to a pair is the product of the distances between the points and the points of the pair. If π_1, π_2, π_3 be the powers of P as to the fundamental pairs, and if α_1, α_2, α_3 be the reciprocals of the half distances of the points of the fundamental pairs, then $\sum(\alpha_i\pi_i)^2 = 0$. If (1) $\lambda x = \alpha_i\pi_i$, then (2) $\sum x_i^2 = 0$. Equations (1) and (2) suffice to determine the x_i as a redundant system of point coordinates. An equation of the form $\sum c_i x_i = 0$ represents a point pair and conversely. Two point pairs ξ_i and η_i are in involution when and only when (3) $\sum \xi_i \eta_i = 0$. If the ξ_i be fixed and the η_i variable, (3) defines an involution ξ_i. The ξ_i are also the coördinates of the focal pair. If ξ_i be variable and η_i fixed, (3) defines a pencil η_i of involutions. The η_i are also the coordinates of the two focal involutions of the pencil. Thus is evidenced the bidimensionality of the range in involutions and in pencils (of involutions) and the dual relationship of these entities as elements. The geometry thus arising in the line (or other like space) is quite as rich as and is algebraically identical with the range-pencil geometry of the plane or any similar doctrine. Ordinarily involution appears as a mere detail of projective geometry. That is because, as ordinarily presented, it is but the intersection of the theory here indicated and the usual projective geometry of the plane. In the geometry here in question, every involution contains one infinite pair (composed of a finite point and the infinite point of the line) and two zero-radial pairs (points of pair falling together). If the ensemble of infinite pairs be taken for absolute, the geometry will be euclidean or parabolic. It will be lobachevskian or hyperbolic if the ensemble of zero-radial pairs be taken for absolute.

14. In Dr. Coble's first paper it is noted that two interchangeable plane collineations of period 3, which are not powers of each other, and which have unique fixed triangles, uniquely

define a plane Hesse configuration. An immediate generalization can be made in spaces of $p-1$ dimensions S_{p-1}, p being any prime number greater than 2. The resulting configuration is made up of p^2 $S_{\frac{1}{2}(p-3)}$'s and p^2 $S_{\frac{1}{2}(p-1)}$'s so situated that the $S_{\frac{1}{2}(p-3)}$'s lie, in sets of p, in $p(p+1)$ S_{p-2}'s; and the $S_{\frac{1}{2}(p-1)}$'s meet, in sets of p, at $p(p+1)$ S_0's. The collineation group associated with the configuration has an invariant subgroup of order $2p^2$, and the factor group is isomorphic with the linear group $LF(2, p)$.

15. Involutory Cremona transformations have been divided by Bertini into four classes according as they are reducible by a Cremona transformation to one or another of four "types." The object of Dr. Coble's second paper is to study the general case of a class by using a construction of the involution which is unaltered by Cremona transformation. In the cases considered the dependence of the fundamental points upon each other is given. The invariant curves and allied line loci are treated in some detail.

16. An algebraic form (homogeneous and isobaric polynomial) in any number of variables x, y, $\cdots$ and in the coefficients of any number of quantics of any orders that is not altered by such a linear transformation as simply adds an arbitrary constant multiple of x to y may be called an xy-differentiant belonging to the system of quantics, and the property just described may be called the differentiant character xy. In 1854 Cayley published a formula for the number of linearly independent xy-differentiants of any given type that belong to a single binary quantic of any order, and this formula was proved and extended to any number of binary quantics by Sylvester in 1877. The object of Professor Story's paper is to extend Cayley's formula to forms in any number of variables and in the coefficients of any number of quantics of any orders that have any *two* given differentiant characters. The denumerants actually given are those for xy- and xz-differentiants, for xz- and yz-differentiants, for xy- and yz-differentiants (which are also xz-differentiants), and for xy- and zy-differentiants, without regard to the whole number of variables involved. The denumerant for in- and covariants of any system of ternary quantics is a particular case of that for xy- and yz-differentiants. Incidentally, a number of properties of differentiants are estab-

lished. The general method of determining denumerants here developed is applicable to any number of differentiant characters, and the writer hopes before long to extend his formulas to triple differentiants, including in- and covariants of quaternary systems.

17. By means of plane sections of ruled surfaces, and of special quadric spreads in hyperspace, Professor Snyder proves that when the points of two nonsingular curves are in (1, 1) correspondence they are projectively equivalent. More generally, if a curve c_n has an i-fold point $(i \leqq n - 3)$ and no other singularity, it cannot be transformed birationally into itself or any other curve of order n except by collineation; similarly, if it have a k-fold point $(k \leqq n - 4)$ and a double point, or not more than $E[(\frac{1}{2}(n-1))^2] - 2$ distinct double points, $E(k)$ being the largest integer in k.

18. In Professor McKinney's paper the axis of reals is divided into intervals of the types $a)$ $a_i - 1 + \lambda \cdots (a_i + \lambda)$, $a_i > 0$; $b)$ $(a_i - \lambda) \cdots a_i + 1 - \lambda$, $a_i < 0$; $c)$ $(a_i - \lambda) \cdots (a_i + \lambda)$, $a_i = 0$, where a_i is an integer, λ a variable parameter, $0 < \lambda \leqq 1$, and the effect of the parentheses is to exclude the inclosed number from the interval. The system of intervals is symmetric with respect to the origin. Every number in the interval is said to correspond to a_i, the representative number of the interval. The defining equation takes the form $x_i = a_i - 1/x_{i+1}$, where x_i, real and not zero, corresponds to a_i.

For all values of λ, $\frac{1}{2}(-1 + \sqrt{5}) < \lambda \leqq 1$, the continued fractions representing two properly equivalent numbers are ultimately alike. For $0 < \lambda \leqq \frac{1}{2}(-1 + \sqrt{5})$ they are ultimately alike except under specified conditions. In the case excepted the continued fraction representing one of the numbers may, from elements of a certain rank on, be readily derived from that representing the other number. The results for improper equivalence follow immediately from the relation of the continued fractions representing x and $-x$.

19. Professor Hawkes's paper gave a proof of the necessary and sufficient conditions for the equivalent of two families of bilinear forms when the determinant of the family is not zero. The proof depends on the reduction of a matrix to normal forms, in

which appear numbers that are precisely the exponents of the elementary divisors of the characteristic determinant of the matrix.

20. Professor Wilson places the theory of rotations in connection with the general theory of matrices or dyadics from the point of view of Gibbs. This leads to an easy and immediate treatment of such questions as those treated in the case of four dimensions by F. N. Cole, *American Journal of Mathematics*, volume 12 (1890). The treatment given by Professor Wilson leads to the general classification and reduction of n-dimensional pairs in case the matter of reality is regarded as important.

21. Professor Kasner's first paper appears in the present number of the BULLETIN.

22. In recent papers published in the *Transactions* Professor Kasner has investigated the characteristic geometric properties of the system of trajectories connected with any positional field of force. In the present paper the particle is acted upon by such a force and also by a resisting medium. Distinct discussions are necessary for the two-dimensional and three-dimensional problems. In both discussions the case where the resistance varies as the square of the velocity is especially interesting since then and only then it turns out that certain of the properties which hold for free motion hold also in the resisted motion. These are the properties numbered I and II in the papers referred to above.

23. Dr. Stephens showed that, if a point t of a circle be projected at a fixed inclination upon the n Wallace lines which arise from n points t_i of the same circle taken $n-1$ at a time, then these n projections lie in a straight line whose envelope, as t varies, is a curve of class n and order $2(n-1)$. If the angle of projection be allowed to vary, there is formed a system of curves whose centers lie on a line. If $n = 3$, Dr. Converse's case results at once; if the points t_i form a regular polygon, we have the case considered by Professor Steggall.

24. If the vanishing of the first variation of two known integrals each containing one dependent variable gives the same differential equation, then by a theorem of Darboux it is possible to write immediately an intermediary integral of this

differential equation. The paper of Dr. Gillespie shows that if two known integrals, each containing n dependent variables, give through the vanishing of their first variation the same system of differential equations, it is then possible to write immediately an integral of this system of differential equations.

25. The general circulation of the atmosphere has been discussed by Ferrel and Oberbeck. The former made no attempt at a complete solution. The latter assumed that the air was incompressible but satisfied Boyle's law. He also surrounded the atmosphere with a spherical boundary of indefinite height. In Mr. Sharpe's solution it is shown that when there is no inequality of temperature between the equator and the poles the air is in both adiabatic and conductive equilibrium. Consequently when there is an increase of temperature from the pole to the equator the temperature still satisfies Fourier's law of conduction and a circulation exists which vanishes at the upper limit of the atmosphere as well as on the earth's surface. This meridional circulation modifies the east and west motion of the air, which in turn changes the pressure distribution and the circulation.

F. N. COLE,
Secretary.

THE DECEMBER MEETING OF THE CHICAGO SECTION.

THE twentieth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago, Chicago, Ill., on Friday, December 28, 1906. The total attendance was thirty-five, including the following twenty-nine members of the Society:

Mr. G. D. Birkhoff, Professor G. A. Bliss, Professor Oskar Bolza, Professor D. R. Curtiss, Professor L. E. Dickson, Dr. Otto Dunkel, Professor T. F. Holgate, Mr. Louis Ingold, Professor O. D. Kellogg, Professor Kurt Laves, Mr. N. J. Lennes, Dr. A. C. Lunn, Mr. W. S. MacMillan, Professor H. Maschke, Professor G. A. Miller, Professor E. H. Moore, Dr. J. C. Morehead, Professor F. R. Moulton, Dr. L. I. Neikirk, Professor H. L. Rietz, Professor N. C. Riggs, Mr. W. J. Risley, Mr. A. R. Schweitzer, Professor J. B. Shaw, Dr. C. H.

Sisam, Professor H. E. Slaught, Mr. R. E. Wilson, Professor J. W. A. Young, Professor Alexander Ziwet.

Two sessions were held for the reading of papers, at both of which the chairman of the Section, Professor Alexander Ziwet, presided. In the evening seventeen members of the Section dined together and enjoyed a season of social intercourse.

The committee having in charge the arrangement of a meeting for the discussion of mathematics for engineers reported that the proposed plans had been modified on account of the conflict in time with the meeting of the American association for the advancement of science in New York. In view of the announcement that this association is to meet in Chicago in December, 1907, the committee was instructed to arrange if possible a joint session of the Chicago Section with Sections A and D of the American association and with the Society for the promotion of engineering education, which meets at the same time and place.

Professor E. B. Van Vleck was elected Chairman and Professor H. E. Slaught Secretary of the Section, for the ensuing year. These, together with Professor H. L. Rietz, constitute the programme committee; and, with the further addition of Professors A. G. Hall, E. J. Townsend, E. B. Skinner and Alexander Ziwet, constitute the committee on programme for the December, 1907, meeting.

The following papers were read:

(1) Professor JAMES B. SHAW: "On the equivalence of algebras."

(2) Professor JAMES B. SHAW: "Quaternion associative lineolinear expressions."

(3) Professor O. BOLZA: "On Lagrange's method of multipliers in the calculus of variations."

(4) Dr. C. H. SISAM: "On a locus associated with plane curves."

(5) Professor G. A. MILLER: "The groups generated by two operators such that each is transformed into its inverse by the square of the other."

(6) Mr. A. R. SCHWEITZER: "On an interesting class of monotonic functions."

(7) Professor L. E. DICKSON: "Invariantive theory of quadratic forms in a finite field."

(8) Professor D. R. CURTISS: "Sufficient conditions for the linear dependence of functions of more than one variable."

(9) Professor G. A. Bliss: "Note on maxima and minima of functions of several variables."

(10) Professor O. D. Kellogg: "The Green formula

$$u(x, y) = \frac{1}{2\pi} \int_C u(\sigma) \frac{\partial}{\partial \nu} G(x, y; \xi, \eta) d\sigma."$$

(11) Dr. A. C. Lunn: "An example of a mechanical system with numerably infinite degrees of freedom."

Abstracts of the papers follow below, the numbers corresponding to the titles in the list above.

1. This first paper of Professor Shaw defines the equivalence of two algebras, thus: Two algebras are equivalent when, if they are supposed to have the same fundamental units, and if the product of two numbers α and β is indicated for one algebra by $\Phi\ \alpha\beta$, for the other by $\Phi'\ \alpha\beta$, then

$$\Phi' \cdot \alpha\beta = \phi\Phi\phi^{-1}\alpha\phi^{-1}\beta,$$

where ϕ is any linear operator which transforms the units $e_1, \cdots, e_r$ into numbers ϵ given by the equations

$$e_i = \phi\epsilon_i = \sum_{j=1}^{r} c_{ij}\epsilon_j \qquad (i = 1, \cdots, r).$$

It is understood that $|\phi| = |c_{ij}| \neq 0$, that is, ϕ has no zero roots.

The consequences of this definition are developed, with some consideration of the two problems: (1) to simplify the defining equations of an algebra by removing unessential parameters; (2) to determine algebras whose form is in some degree invariant under the transformation ϕ, these algebras furnishing means of defining other algebras without reference to a multiplication table.

2. The second paper of Professor Shaw develops all lineo-linear functions of two quaternions q, r which are associative, that is, for which

$$F(q, F[r, s]) = F(F[q, r], s).$$

3. Professor Bolza's paper gives a proof of Lagrange's

rule of multipliers and of the limit equations for the problem of minimizing the integral

$$I=\int_{t_0}^{t_1} f(y_1, y_2, \cdots, y_n; y'_1, y'_2, \cdots, y'_n)dt$$

when the admissible curves are subject to the condition of satisfying p differential equations

$$\phi_\alpha(y_1, y_2, \cdots, y_n; y'_1, y'_2, \cdots, y'_n) = 0 \quad (\alpha = 1, 2, \cdots, p)$$

and q finite equations

$$\psi_\beta(y_1, y_2, \cdots, y_n) = 0 \quad (\beta = 1, 2, \cdots, q),$$

while the coördinates of their end points $(y_{10}, y_{20}, \cdots, y_{n0})$, $(y_{11}, y_{21}, \cdots, y_{n1})$ satisfy r relations of the form

$$\chi_\gamma(y_{10}, y_{20}, \cdots, y_{n0}, y_{11}, y_{21}, \cdots, y_{n1}) = 0 \quad (\gamma = 1, 2, \cdots, r).$$

The proof is based upon an extension of the method used by Hilbert for the special case when $q = 0$ and the end points are fixed. The paper will appear in the *Mathematische Annalen.*

4. Dr. Sisam's paper treats of the locus of the points of intersection of the tangents to an arbitrary algebraic curve whose points of tangency lie on a common tangent to a second algebraic curve. The order, and other interesting properties of this locus, and the corresponding properties of the dual configuration are derived. Two particular cases, first when the second of the given curves reduces to a point, and second when the locus curve coincides with the first of the two given curves, are studied in detail.

5. The main theorems proved in Professor Miller's paper may be stated as follows: If two operators neither of which is identity are such that each is transformed into its inverse by the square of the other, they must have the same order and this common order is 2, 4 or 8. If the common order exceeds two, the group generated by them involves operators of order 3; in fact, this group must have an $(\alpha, 1)$ isomorphism with the symmetric group of order 24. The smallest possible group which is generated by two such operators of order 8 is the group of order 48 known as G_{52}, and every other group that

can be generated by two such operators has an $(\alpha, 1)$ isomorphism with this group of order 48. The paper also contains a new and useful definition of the symmetric group of order 24. This paper is to be offered to the *Annals of Mathematics* for publication.

6. Representing the points $0 \leqq x \leqq 1$ by the series

$$x = \sum_{k=1}^{+\infty} \frac{\alpha_k}{m_1 m_2 \cdots m_k},$$

where the numbers α_k, m_k are integers, $0 \leqq \alpha_k \leqq m_k - 1$, $m_k \geqq 2$, Mr. Schweitzer discusses the function

$$t(x) = \sum_{k=1}^{+\infty} r_k \frac{\alpha_k^{l_k}}{m_1' m_2' \cdots m_k'},$$

where l_k, m_k' are integers. The paper will be offered to the *Annals of Mathematics* for publication.

7. In the paper by Professor Dickson there is established a theorem on m-ary quadratic forms in the $GF[2^n]$ analogous to that concerning the rank r of the discriminantal determinant Δ in the algebraic theory and the minimum number r of variables on which the form can be expressed. In place of Δ (for m odd) and of all minors of odd order, we employ the halves of their algebraic expansions, necessarily with even coefficients. In particular, the semi-discriminant is an invariant. When the rank r is even, there are two classes of forms, distinguishable by an arithmetic property of Δ^{2^n-1}. Following this preliminary part, the paper proceeds to the determination and interpretation of a complete set of linearly independent invariants of the quadratic form. This invariantive theory differs in essential features from the simple algebraic theory. The paper will be offered for publication to the *American Journal of Mathematics*.

8. If $u_1, u_2, \cdots, u_m$ are m functions of n variables all of whose partial derivatives of the first $m - 1$ orders exist and are finite throughout a given region, a necessary condition for their linear dependence is the vanishing of all m-rowed determinants of the matrix whose first row consists of the m functions u, and whose succeeding rows are composed of partial derivatives of the elements of the first row, so that to each deriv-

ative of the first $m - 1$ orders there corresponds a row. In Professor Curtiss's paper some properties of matrices of this sort are developed and sufficient conditions for the linear dependence of the functions u are given.

9. In investigating whether or not a function $f(x_1, x_2, \cdots, x_n)$ has a maximum or minimum at a point $(a_1, a_2, \cdots, a_n)$, it is usual to suppose that all the derivatives of the function of order less than n vanish at the point $(a_1, a_2, \cdots, a_n)$, while one at least of order n is different from zero. Then the function is expanded by Taylor's formula with a remainder to terms of order $n + 1$, or else in an infinite series. In Professor Bliss's paper it is shown how the usual results in the definite and indefinite cases can be obtained from Taylor's expansion to terms of order n only. The use of derivatives of higher orders than those occurring in the criteria which enable one to decide if there is a maximum or minimum is thus avoided. This paper will be offered to the BULLETIN for publication.

10. If $u(x, y)$ be a potential function of a simply connected closed region R of the (x, y)-plane bounded by a regular closed curve C, then it is known that if u and its first derivatives are continuous on the boundary of R, it may be represented in the form

$$u(x, y) = \frac{1}{2\pi}\int_C u(\sigma)\frac{\partial}{\partial\nu}G(x, y;\ \xi, \eta)d\sigma,$$

exhibiting the dependence of the function upon its boundary values $u(\sigma)$. Starting on the other hand from this formula, the question arises: under what conditions upon boundary curve C and upon boundary values $u(\sigma)$ is the above integral a potential function of R which approaches the boundary values $u(\sigma)$? For regions bounded by curves the coordinates of whose points as functions $x(s)$, $y(s)$ of the length of arc have uniformly "algebraically continuous" derivatives (*i. e.*, $|\Delta x'(s)| < A|\Delta s|^\alpha$, $|\Delta y'(s)| < A|\Delta s|^\alpha$, $0 < \alpha$ so soon as $|\Delta s| < \delta$), Professor Kellogg shows that a potential function is determined by a given set of boundary values through the above formula under the same conditions upon $u(\sigma)$ as in the case of the circle through Poisson's integral.

11. Dr. Lunn's paper gives the solution of an infinite system of ordinary differential equations, which are shown to determine

the motion of an infinite linear system of discrete masses, connected by springs. The solution is obtained indirectly by a limiting process from the solution for a finite number of masses, and is then verified directly. The main features of the oscillations of a given mass are interpreted in terms of familiar properties of the Bessel functions of the time which occur as coefficients.

H. E. SLAUGHT,
Secretary of the Section.

THE DECOMPOSITION OF MODULAR SYSTEMS CONNECTED WITH THE DOUBLY GENERALIZED FERMAT THEOREM.

BY PROFESSOR ELIAKIM HASTINGS MOORE.

(Read before the Chicago Section of the American Mathematical Society, December 29, 1898.)

Introduction. The Generalized Fermat Theorem (A) *in Purely Arithmetic Phrasing* (A', A'') *with Extension* (A'''). §§ 1–5.

1. The theorem * in question is the following:

(A) In the Galois field $GF[p^n]$ of prime modulus p and of rank n the two forms each of degree $(p^{n(k+1)}-1)/(p^n-1)$ in the $k+1$ indeterminates $X_0, X_1, \cdots, X_k$

$$D_{k+1,n,p}[X_0, X_1, \cdots, X_k] = | X_j^{p^{ni}} | \qquad (i, j = 0, 1, \cdots, k)$$

$$P_{k+1,n,p}[X_0, X_1, \cdots, X_k] = \prod_{g=0,k} \prod_{a_{fg}|p^n} (X_g + \sum_{f=0,g-1} a_{fg} X_f)$$

are identical:

$$D_{k+1,n,p}[X_0, \cdots, X_k] = P_{k+1,n,p}[X_0, \cdots, X_k].$$

Here the subscript remark $a_{fg}|p^n$ indicates that the mark a_{fg} is to run over the p^n marks of the Galois field $GF[p^n]$, and for the case $g=0$ the final $\sum_{f=0,g-1}$ does not enter.

For this theorem, which for $(k, n) = (1, 1)$ is one form of Fermat's theorem, I have given three proofs, couched as is the statement of the theorem in the abstract Galois field phrasing introduced by me in the paper "A doubly-infinite system of simple groups" presented to the Chicago Congress of 1893.

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* Moore, "A two-fold generalization of Fermat's theorem," BULLETIN, vol. 2 (1896), pp. 189–199.

In the development of the Galois field theory and in its applications to algebra and groups this abstract phrasing is very convenient.

2. Here however we are interested in converses of the theorem stated above; after replacing the marks α_{fg} by functions of the realm $[1, y]$ we decompose, in the sense of equivalence, into prime modular systems the modular system $\mathfrak{M}$ whose elements are the coefficients of various powers of the indeterminates in the form

$$M_{k+1, n, p} \equiv D_{k+1, n, p} - P_{k+1, n, p}.$$

We need first to use the concrete purely arithmetic phrasings of Serret and Kronecker. The $GF[p^n]$ is then the totality of p^n classes (abstractly, marks) of rational integral functions of an indeterminate y with integral coefficients (forming the realm of integrity $[1, y]$) considered with respect to a (prime) modular system $[p, F_n[y]]$, where $F_n[y]$ is a function of the realm $[1, y]$ of degree n (a polynomial in y with integral coefficients) irreducible modulo p. We operate with these classes by operating with representative functions with respect to the modular system $[p, F_n[y]]$. For the marks α we may take the p^n reduced functions

$$a_0 + a_1 y + a_2 y^2 + \cdots + a_{n-1} y^{n-1},$$

where the n coefficients $a_0, \cdots, a_{n-1}$ take independently the values $0, 1, 2, \cdots, p - 1$. There are in this sense as many concretely distinct Galois fields as there are functions $F_n[y]$ congruentially distinct modulo p. These Galois fields are however abstractly identical.

We restate theorem

(A') In the realm of integrity $[1, y]$ the two forms each of degree $(p^{n(k+1)} - 1)/(p^n - 1)$ in the $k + 1$ indeterminates $X_0, \cdots, X_k$

$$D_{k+1, n, p}[X_0, \cdots, X_k] = |X_j^{p^{ni}}| \quad (i, j = 0, 1, \cdots, k)$$

$$P_{k+1, n, p}[X_0, \cdots, X_k]$$
$$= \prod_{g=0, k} \prod_{a_{fgl}=0, p-1} (X_g + \sum_{f=0, g-1} X_f \sum_{l=0, n-1} a_{fgl} y^l)$$

are identically * congruent ($\equiv$):

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* In pure arithmetic all identities and identical congruences are *formal* in certain specified indeterminates.

$$D_{k+1,n,p}[X_0, \cdots, X_k] \equiv P_{k+1,n,p}[X_0, \cdots, X_k] \quad [p, F_n[y]],$$

with respect to every prime modular system $[p, F_n[y]]$, where p is a prime and $F_n[y]$ is any function of $[1, y]$ of degree n and irreducible modulo p.

3. I take as known the fundamental definitions and elementary theorems of Kronecker's theory of modular systems, in particular with respect to the composition and equivalence of modular systems. It is however desirable to fix the notations * to be used here.

A realm $\mathfrak{R}$ of integrity-rationality

$$\mathfrak{R} = [R_1 \cdots, R_\mu](R_{\mu+1}, \cdots, R_{\mu+\nu})$$

consists of all functions

$$F[R_1, \cdots, R_\mu](R_{\mu+1}, \cdots, R_{\mu+\nu})$$

rational integral in $R_1, \cdots, R_\mu$ and rational in $R_{\mu+1}, \cdots, R_{\mu+\nu}$, the coefficients being integers. These functions are called the *quantities* of the realm. The realm is closed under addition, subtraction, and multiplication, and likewise under division by any function not 0 of $\mathfrak{R}' = (R_{\mu+1}, \cdots, R_{\mu+\nu})$.

Any set of quantities $F_1, \cdots, F_m$ of a realm $\mathfrak{R}$ constitutes a modular system $\mathfrak{F} = [F_1, \cdots, F_m]$ of that realm. The whole theory of such modular systems relates to the underlying realm.

Any set of modular systems

$$\mathfrak{F}_i = [F_{i1}, \cdots, F_{im_i}] \qquad (i = 1, 2, \cdots, m)$$

determines a modular system $[\cdots F_{ij_i} \cdots_{\substack{i=1,\cdots,m \\ j_i=1,\cdots,m_i}}]$ for which we use the notation $[\mathfrak{F}_1, \cdots, \mathfrak{F}_m]$.

(1) The theorem:

If

$$[\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}] \sim [1],$$

then

$$[\mathfrak{F}_1, \mathfrak{F}][\mathfrak{F}_2, \mathfrak{F}] \sim [\mathfrak{F}_1\mathfrak{F}_2, \mathfrak{F}],$$

and its useful generalization:

If

$$[\mathfrak{F}_i, \mathfrak{F}_j, \mathfrak{F}] \sim [1] \qquad (i \neq j;\ i, j = 1, \cdots, m),$$

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* The notations are those of my paper "The decomposition of modular systems of rank n in n variables," BULLETIN, vol. 3 (1897), pp. 372–380.

then

$$\prod_{i=1,m}[\mathfrak{F}_i, \mathfrak{F}] \sim [\prod_{i=1,m}\mathfrak{F}_i, \mathfrak{F}].$$

may readily be proved.

(2) A system $\mathfrak{F}$ of the realm $\mathfrak{R}$ I call *prime* * in case (α) $\mathfrak{F}$ and $[1]$ are not equivalent and (β) for any quantity G of the realm we have one of the alternative equivalences

$$[\mathfrak{F}, G] \sim \mathfrak{F} \quad \text{or} \quad [\mathfrak{F}, G] \sim [1],$$

that is, one of the alternative congruences

$$G \equiv 0[\mathfrak{F}] \quad \text{or} \quad GG' \equiv 1\ [\mathfrak{F}],$$

where in the latter case G' is a properly determined quantity of the realm.

For a prime system $\mathfrak{F}$ in connection with any system $\mathfrak{G}$ we have one of the alternatives

$$[\mathfrak{F}, \mathfrak{G}] \sim \mathfrak{F} \quad \text{and} \quad \mathfrak{G} \equiv 0[\mathfrak{F}] \quad \text{or} \quad [\mathfrak{F}, \mathfrak{G}] \sim [1],$$

that is, any system $\mathfrak{G}$ contains or is relatively prime to a prime system $\mathfrak{F}$.

We have the fundamental theorem:

If $\mathfrak{F}$ is prime and the product $\mathfrak{G}_1\mathfrak{G}_2$ of two systems $\mathfrak{G}_1$, $\mathfrak{G}_2$ contains $\mathfrak{F}$, while $\mathfrak{G}_1$ does not contain $\mathfrak{F}$, then $\mathfrak{G}_2$ does contain $\mathfrak{F}$.

For $\mathfrak{G}_1\mathfrak{G}_2 \equiv 0\ [\mathfrak{F}]$ and $\mathfrak{G}_1 \not\equiv 0\ [\mathfrak{F}]$ and hence, since $\mathfrak{F}$ is prime, $[\mathfrak{F}, \mathfrak{G}_1] \sim [1]$. Now $\mathfrak{F}\mathfrak{G}_2 \equiv 0\ [\mathfrak{F}]$. Hence we have $[\mathfrak{F}, \mathfrak{G}_1]\mathfrak{G}_2 \equiv 0\ [\mathfrak{F}]$, and so indeed $\mathfrak{G}_2 \equiv 0\ [\mathfrak{F}]$.

(3) If $[\mathfrak{F}, \mathfrak{G}_i] \sim [1]$ $(i = 1, 2, \cdots, m)$, then

$$[\mathfrak{F}, \prod_{i=1,m}\mathfrak{G}_i] \sim \prod_{i=1,m}[\mathfrak{F}, \mathfrak{G}_i] \sim [1].$$

(4) If $[\mathfrak{F}_i, \mathfrak{F}_j] \sim [1]$ $(i \neq j;\ i, j = 1, 2, \cdots, m)$ and $\mathfrak{M} \equiv 0\ [\mathfrak{F}_i]$ $(i = 1, 2, \cdots, m)$, then $\mathfrak{M} \equiv 0\ [\mathfrak{F}]$ where $\mathfrak{F} = \prod_i \mathfrak{F}_i$.

4. We are now to work in the realm of integrity $\mathfrak{R}_1 = [1, y]$, where y is an indeterminate. There are in all say m functions $F_n[y]$ of $\mathfrak{R}_1$ congruentially distinct (mod. p) and each of degree n and irreducible mod. p; we write them $F_{nj}[y]$ $(j = 1, \cdots, m)$. The modular systems $\mathfrak{F}_{nj} = [p, F_{nj}[y]]$ $(j = 1, \cdots, m)$ are

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* Kronecker's prime modular system of a given rank (*Jour. für Mathematik*, vol. 99, p. 337) is differently defined.

all prime and by pairs relatively prime. We write their product

$$\mathfrak{F}_n = \prod_{j=1,m} \mathfrak{F}_{nj} = \prod_{j=1,m} [p, F_{nj}[y]] \sim [p, \prod_{j=1,m} F_{nj}[y]].$$

5. We set (using the notations of § 2)

$$M_{k+1,n,p}[X_0, \cdots, X_k] = D_{k+1,n,p}[X_0, \cdots, X_k] \\ -P_{k+1,n,p}[X_0, \cdots, X_k]$$

and denote by $\mathfrak{M}_{k+1,n,p}$ the modular system of coefficients of $M_{k+1,n,p}[X_0, \cdots, X_k]$; this system $\mathfrak{M}$ belongs to the realm $\mathfrak{R}_1 = [1, y]$.

Our theorem A'(§ 2) is then

$$(A'') \qquad \mathfrak{R}_{k+1,n,p} \equiv 0 \; [\mathfrak{F}_{nj}] \qquad (j = 1, \cdots, m),$$

whence follows in accordance with § 4 and § 3 (4),

$$(A''') \qquad \mathfrak{M}_{k+1,n,p} \equiv 0 \; [\mathfrak{F}_n].$$

The Equivalence Theorem (B) *and the Decomposition Theorem* (C) §§ 6–14.

6. We are to prove the equivalence

$$(B) \qquad \mathfrak{M}_{k+1,n,p} \sim \mathfrak{F}_n,$$

whence in view of § 4 follows the decomposition, in the sense of equivalence, of the system $\mathfrak{M}_{k+1,n,p}$, viz.,

$$(C) \qquad \mathfrak{M}_{k+1,n,p} \sim \prod_{j=1,m} \mathfrak{F}_{nj},$$

that is, its exhibition as a product of systems no further decomposable and indeed in this case prime.

7. To prove (B) we need to prove, (A''') being admitted, merely the converse of (A'''), viz.,

$$(\bar{A}''') \qquad \mathfrak{F}_n \equiv 0 \quad [\mathfrak{M}_{k+1,n,p}].$$

Here $\mathfrak{F}_n$ depends upon n and p and not on k. The congruence $(\bar{A}''')$ follows from the following two congruences:

$$(\bar{A}'''_{k=1}) \qquad \mathfrak{F}_n \equiv 0 \quad [\mathfrak{M}_{2,n,p}],$$

$$(D) \qquad \mathfrak{M}_{2,n,p} \equiv 0 \quad [\mathfrak{M}_{k+1,n,p}],$$

of which the former is the particular case of $(\bar{A}''')$ for $k = 1$. We shall, however, prove (D) and then (B, C) without the mediation of $(\bar{A}'''_{k=1})$.

8. Proof of (D). The obviously true congruence

$$\mathfrak{M}_{k+1,n,p} \equiv 0 \quad [\mathfrak{M}_{k+1,n,p}]$$

implies the identical congruence

$$M_{k+1,n,p}[X_1, X_1, \cdots, X_0] \equiv 0 \quad [\mathfrak{M}_{k+1,n,p}]$$

in the $k+1$ indeterminates $X_0, \cdots, X_k$. Considering in particular the terms of M with the literal part containing the factor $X_2^{p^{2n}} \cdots X_i^{p^{in}} \cdots X_k^{p^{kn}}$, we have the identical congruence in X_0, X_1

$$(X_0 X_1^{p^n} - X_1 X_0^{p^n}) -$$
$$\prod_{\substack{a_0 \cdots a_{n-1} \\ =0,\, p-1}} (X_1 + X_0 \overline{a_0 + a_1 y + \cdots + a y^l + \cdots + a_{n-1} y^{n-1}}) \equiv 0 \quad [\mathfrak{M}_{k+1,n,p}],$$

that is,

$$M_{2,n,p}[X_0, X_1] \equiv 0 \quad [\mathfrak{M}_{k+1,n,p}],$$

and so in fact we have

$$(D) \qquad \mathfrak{M}_{2,n,p} \equiv 0 \quad [\mathfrak{M}_{k+1,n,p}].$$

9. We consider in preparation for the proof of (B, C) (§ 11) various properties of the system $\mathfrak{M} = \mathfrak{M}_{k+1,n,p}$, consequences of the identical congruence

$$M[X_0, X_1] = D[X_0, X_1] - P[X_0, X_1] = (X_0 X_1^{p^n} - X_1 X_0^{p^n})$$
$$- \prod_{\substack{a_0 \cdots a_{n-1} \\ =0,\, p-1}} (X_1 + X_0 \overline{a_0 + a_1 y + \cdots + a_l y^l + \cdots + a_{n-1} y^{n-1}}) \equiv 0 \, [\mathfrak{M}];$$

this congruence obviously true for $\mathfrak{M} = \mathfrak{M}_{2,n,p}$ is by (D) true for $\mathfrak{M} = \mathfrak{M}_{k+1,n,p}$.

(1) G being any quantity $G = a_0 + a_1 y + \cdots + a_{n-1} y^{n-1}$ of our realm $\mathfrak{R}_1 = [1, y]$ of degree in y at most $n - 1$ with coefficients a_l each 0 or positive integers less than p but not all 0, we have

$$[\mathfrak{M}, G] \sim [1],$$

and so there is a quantity G' of $\mathfrak{R}_1$ such that

$$GG' \equiv 1 \quad [\mathfrak{M}].$$

Proof. Since $\mathfrak{M} \equiv 0 \quad [\mathfrak{M}, G]$ we have

$$D[X_0, X_1] \equiv P[X_0, X_1] \quad [\mathfrak{M}, G].$$

Now $P[X_0, X_1]$ has the distinct factors X_1 and $X_1 + GX_0$ and other factors with product say $Q[X_0, X_1]$; hence

$$P[X_0, X_1] \equiv X_1^2 Q[X_0, X_1] \quad [G]$$

and so, since $G \equiv 0 [\mathfrak{M}, G]$,

$$P[X_0, X_1] \equiv X_1^2 Q[X_0, X_1] \quad [\mathfrak{M}, G].$$

Hence

$$D[X_0, X_1] \equiv X_1^2 Q[X_0, X_1] \quad [\mathfrak{M}, G].$$

The term $X_1 X_0^{p^n}$ gives the congruence

$$-1 \equiv 0 \quad [\mathfrak{M}, G],$$

that is, $[1] \equiv 0 [\mathfrak{M}, G]$, and, since $[\mathfrak{M}, G] \equiv 0$ $[1]$, we have proved that $(\mathfrak{M}, G] \sim [1]$.

(2°) There is an integer g for which

$$pg \equiv 0 \quad [\mathfrak{M}] \quad \text{and} \quad [p, g] \sim [1].$$

Proof. From the identical congruence by the substitution $(X_0, X_1) = (1, -\overline{p-1})$ we have a congruence identical in y, and the term y° gives the congruence

$$(-\overline{p-1})^{p^n} + \overline{p-1} \equiv 0 \quad [\mathfrak{M}],$$

whence follows the statement (2) in case $p = 2$ for $g = 1$ at once, and in case $p > 2$ for $g = (\overline{p-1}^{p^n} - \overline{p-1})/p$ by the remark that $pg \equiv -p$ $[p^2]$ and so $g \equiv -1$ $[p]$.

$$(3) \qquad p^{p^{n-1}} \equiv 0 \quad [\mathfrak{M}].$$

Proof. In case $p = 2$ (3) follows at once from (2), $2 \equiv 0$ $[\mathfrak{M}]$, as proved for $p = 2$. In case $p > 2$ the coefficient of $X_0^2 X_1^{p^n - 1}$ gives

$$[\tfrac{1}{2}p(p-1)]^{p^{n-1}} \cdot (1 + y + \cdots + y^{n-1}) \equiv 0 \quad [\mathfrak{M}].$$

Setting $G_1 = \frac{1}{2}(p-1)$ and $G_2 = 1 + \cdots + y^{n-1}$ we have (1) quantities G'_1, G'_2 of $\mathfrak{R}_1$ for which $G_1 G'_1 \equiv 1$, $G_2 G'_2 \equiv 1$ $[\mathfrak{M}]$, and obtain (3) in multiplying the congruence just written by $G_1'^{p^{n}-1} G'_2$.

$$p \equiv 0 \quad [\mathfrak{M}]. \tag{4}$$

Proof. From (2), $[p, g] \sim [1]$, it follows by § 3 (3) that $[p^{p^{n}-1}, g] \sim [1]$, that is, for properly determined g', $gg' \equiv 1$ $[p^{p^{n}-1}]$, so that by (3) $gg' \equiv 1$ $[\mathfrak{M}]$, whence from (2), $pg \equiv 0$ $[\mathfrak{M}]$ we have the desired congruence (4).

$$y^{p^n} - y \equiv 0 \quad [\mathfrak{M}]. \tag{5}$$

Proof. (5) follows from the identical congruence by the substitution $(X_0, X_1) = (1, y)$ with the remark that, since $P[X_0, X_1]$ has the factor $X_1 + \overline{p-1}\,yX_0$, $P[1, y] \equiv 0$ $[p]$ and so by (4) $P[1, y] \equiv 0$ $[\mathfrak{M}]$.

10. We need further the known decomposition

$$[p, y^{p^n} - y] \sim \prod_{d} \prod_{j=1, m_d} [p, F_{dj}[y]],$$

where for every divisor d of n the $F_{dj}[y]$ $(j = 1, \cdots, m_d)$ are the say m_d quantities of $\mathfrak{R}_1$ congruentially distinct and each of degree d and irreducible modulo p. We may and do suppose that the coefficients of the $F_{dj}[y]$ are taken from the integers $0, 1, \cdots, p-1$. Here the $F_{nj}[y]$ are the $F_{nj}[y]$ of § 4; $m_n = m$. The systems $[p, F_{dj}[y]]$ are by pairs relatively prime.

11. Proof of (B, C): $\mathfrak{M} \sim \mathfrak{F}_n \sim \prod_{j=1, m} \mathfrak{F}_{nj}$. We have from § 9 (4, 5)

$$\mathfrak{M} \sim [\mathfrak{M}, p, y^{p^n} - y],$$

whence by § 10 and § 3 (1) we have the decomposition

$$\mathfrak{M} \sim \prod_{d} \prod_{j=1, m_d} [\mathfrak{M}, p, F_{dj}[y]],$$

and so, since $[\mathfrak{M}, F_{dj}[y]] \sim [1]$ for $d < n$ (§ 9, 1),

$$\mathfrak{M} \sim \prod_{j=1, m} \{\mathfrak{M}, p, F_{nj}[y]\},$$

that is, in the notation of § 4,

$$\mathfrak{M} \sim \prod_{j=1,m} [\mathfrak{M}, \mathfrak{F}_{nj}].$$

Now by (A'') $\mathfrak{M} \equiv 0$ $[\mathfrak{F}_{nj}]$ and so $[\mathfrak{M}, \mathfrak{F}_{nj}] \sim \mathfrak{F}_{nj}$, and thus we have finally

$$(B,C) \quad \mathfrak{F} \sim \prod_{j=1,m} \mathfrak{F}_{nj} \sim \mathfrak{F}_n.$$

12. It is noteworthy that the modular system $\mathfrak{M} = \mathfrak{M}_{k+1,n,p}$ is in the sense of equivalence in fact independent of k and dependent only on n and p,

$$\mathfrak{M}_{k+1,n,p} \sim \mathfrak{F}_n \sim \mathfrak{M}_{2,n,p}.$$

13. My exhibition of the system $\mathfrak{F}_n$ by its equivalent $\mathfrak{M}_{k+1,n,p}$ for k any positive integer, the elements of $\mathfrak{M}_{k-1,n,p}$ being the coefficients of the integral function $M_{k+1,n,p}$, is to be compared with that of Serret (Algèbre supérieure, fifth edition, volume 2, §349). In our notation we have (using §3, 1°)

$$\mathfrak{F}_n \sim \prod_{j=1,m} \mathfrak{F}_{nj} \sim \prod_{j=1,m} [p, F_{nj}[y]] \sim [p, F_n[y]],$$

where

$$F_n[y] = \prod_{j=1,m} F_{nj}[y].$$

Serret gives a fraction $F[y]$

$$\prod_{n_e} (y^{p^{n_e}} - y) / \prod_{n_o} (y^{p^{n_o}} - y),$$

where n_e, n_o run through those divisors of n whose complementary divisors n/n_e, n/n_o have respectively an even (or 0), an odd number of unrepeated prime factors, and shows that modulo p the division indicated by the notation of $F(y)$ can be performed and that for the resulting integral function $F[y]$ we have $F[y] \equiv F_n[y]$ $[p]$ and so $\mathfrak{F}_n \sim [p, F[y]]$. — Apparently my exhibition lends itself more easily to investigations in the domain of pure arithmetic.

THE UNIVERSITY OF CHICAGO.

SYSTEMS OF EXTREMALS IN THE CALCULUS OF VARIATIONS.

BY PROFESSOR EDWARD KASNER.

(Read before the American Mathematical Society, December 29, 1906.)

THE extremals connected with any problem in the calculus of variations are the curves or surfaces or higher manifolds which satisfy the first of the conditions necessary for a maximum or minimum value. In the case of the integral of simplest type

$$\int F(x, y, y')dx \tag{1}$$

this condition is given by the Euler-Lagrange equation

$$\frac{d}{dx}F_{y'} - F_y = 0, \tag{2}$$

where the subscripts indicate partial differentiation. The expanded equation is

$$F_{y'y'}y'' + F_{y'y}y' + F_{y'x} - F_y = 0. \tag{3}$$

The extremals here form a doubly infinite system of plane curves. Darboux's well-known investigation of the inverse problem shows that the system is entirely general*: Any doubly infinite system of curves may be regarded as the system of extremals belonging to some integral of type (1).

In this respect the simplest type of problem is unique. In all the more complicated integrals, involving either derivatives of higher order, or more than one dependent or independent variable, the Euler-Lagrange equations are of special form, and hence the system of extremals (curves or other manifolds) must possess peculiar geometric properties. Our object is to give an example, apparently the simplest, of such a property.

Consider the plane problem of second order connected with the integral

$$\int F(x, y, y', y'')dx. \tag{4}$$

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* Darboux, Théorie des surfaces, vol. 3, no. 604.

The extremals form a quadruply infinite system of curves with the equation

$$(5) \qquad \frac{d^2}{dx^2} F_{y''} - \frac{d}{dx} F_{y'} + F_y = 0.$$

This is found to be of the form

$$(6) \qquad y^{\mathrm{IV}} = Ay'''^2 + By''' + C,$$

where A, B, C involve only x, y, y', y''. It will not be necessary, for our purpose, to have the values of these coefficients in terms of the partial derivatives (of first, second, and third order) of F.

An extremal is determined when the initial values of x, y, y', y'', y''' are assigned. If only x, y, y', y'' are given, that is, a curvature element, there will be ∞^1 corresponding curves. For each of these at the common point construct the osculating conic. The locus of the centers of the conics so obtained may be termed the *central locus* corresponding to the given curvature element. What is the character of the central locus?

The general formulas for the center of the conic osculating a given curve at a given point may be obtained most readily by means of a theorem of Cayley: the center is determined by the intersection of consecutive axes of deviation.* Using coordinates X, Y referring to the given point as origin, we find the equation of the axis of deviation at the given point to be

$$(7) \qquad Y/X = m,$$

where

$$(7') \qquad m = \frac{y'y''' - 3y''^2}{y'''}.$$

At the consecutive point the equation becomes

$$(8) \qquad \frac{Y - dy}{X - dx} = m + m'dx,$$

where

$$(8') \qquad m' = \frac{y''(3y''y^{\mathrm{IV}} - 5y'''^2)}{y'''^2}.$$

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* Salmon-Fiedler, Höhere Kurven, p. 468. The point in question is sometimes termed the center of deviation.

The intersection of (7) and (8) is the required center

$$(9) \qquad X=\frac{3y''y'''}{5y'''^2-3y''y^{\mathrm{IV}}}, \qquad Y=\frac{3y''(y'y'''-3y''^2)}{5y'''^2-3y''y^{\mathrm{IV}}}.$$

To apply these general formulas to the question proposed, we first solve them with respect to y''', y^{IV}, obtaining

$$(10) \qquad y'''=\frac{-3y''^2X}{Y-y'X}, \quad y^{\mathrm{IV}}=\frac{3y''^2(5y''^2X+Y-y'X)}{(Y-y'X)^2}.$$

Substituting these values in (6), we find the relation

$$(11) \qquad \begin{aligned}&3y''^2(Y-y'X)\\ &\quad =3y''^3(3y''A-5)X^2-3y''^2BX(Y-y'X)+C(Y-y'X)^2.\end{aligned}$$

Here A, B, C have fixed values since the initial element x, y, y', y'' is supposed to be given. The central locus, defined by (11), is thus a conic. Furthermore it passes through the given point, the origin of the X, Y system; and its tangent at that point, namely,

$$(12) \qquad Y-y'X=0,$$

has the direction of the given element. We state the result as follows:

The extremals connected with any plane problem of the second order, that is, with any integral of type (4), *form a quadruply infinite system of curves such that* ∞^1 *are determined by a given element of curvature. The locus of the centers of the conics which osculate these* ∞^1 *curves at their common point is a conic tangent to the given element.*

From the discussion it is seen that the result is not completely characteristic since it holds for all systems whose differential equation is of form (6), and such a system need not be a system of extremals. In the latter case the functions A, B, C are connected by rather complicated relations derived by expressing the fact that certain partial differential equations are consistent.

That the property stated is, however, characteristic of the form (6) may be shown as follows. By assumption, the central locus for any element of curvature is to be a conic touching the element. Its equation is then

$$Y - y'X = aX^2 + bXY + cY^2. \tag{13}$$

Substituting the general values given in (9), and reducing, we find that y^{IV} is quadratic in y'''. The equation is therefore of the form (6).

If it is required merely that the central locus shall be a conic, without any restriction as to position, then a more general class of differential equations of the fourth order is obtained, involving five instead of three arbitrary functions of x, y, y', y''. In particular, if the conic is to pass through the given point, the resulting equation is of the form

$$y^{\mathrm{IV}} = \alpha y'''^3 + \beta y'''^2 + \gamma y''' + \delta. \tag{14}$$

An interesting special case of systems of type (6) is obtained by imposing the condition that for every curvature element the conic (11) shall degenerate. This is so only when A has the special form

$$A = \frac{5}{3y''}. \tag{15}$$

The conic then breaks up into the trivial part (12) and the proper part

$$3y''^2(BX + 1) - C(Y - y'X) = 0. \tag{16}$$

In the case of a system of extremals the value of A in terms of the integrand function F is

$$A = -\frac{F_{y''y''y''}}{F_{y''y''}}. \tag{17}$$

By comparing (15) and (17) and integrating, we find

$$F = f_1(x, y, y')y''^{\frac{1}{3}} + f_2(x, y, y')y'' + f_3(x, y, y'), \tag{18}$$

where the f's are arbitrary functions.

Hence the only case in which the system of extremals connected with an integral (4) *is such that the central loci are straight lines is that in which the integrand function is of the form* (18).

COLUMBIA UNIVERSITY.

A NECESSARY CONDITION FOR AN EXTREMUM OF A DOUBLE INTEGRAL.

BY PROFESSOR MAX MASON.

(Read before the American Mathematical Society, April 28, 1906.)

LEGENDRE'S necessary condition on a function $y = \phi(x)$ which minimizes or maximizes the integral

$$\int_{x_0}^{x_1} f(x, y, y')dx$$

is that the function $f_{y'y'}(x, \phi, \phi')$ must be positive or zero for a minimum, negative or zero for a maximum. There seems to have been no adequate treatment of double integrals in regard to the analogous condition.

Suppose the function $z = \phi(x, y)$ is of class* D' in a region Ω, assumes the values g on the boundary of Ω, and renders the integral

$$J = \int\int_{\Omega} f(x, y, z, p, q)dxdy \quad (p = z_x, q = z_y)$$

an extremum with respect to all functions of class D' which assume the values g on the boundary. Let $\zeta(x, y)$ be an allowed variation, *i. e.*, a function of class D' which vanishes on the boundary. Then the value of the integral

$$J(\epsilon) = \int\int_{\Omega} f(x, y, \phi + \epsilon\zeta, \phi_x + \epsilon\zeta_x, \phi_y + \epsilon\zeta_y)dxdy,$$

where ϵ is a parameter, is a function of ϵ which has an extreme value for $\epsilon = 0$. It follows† that $J'(0)$ must vanish, and that $J''(0)$ must have the same sign or vanish, for all allowed variations ζ. From the condition $J'(0) = 0$ is derived the differential equation which ϕ must satisfy. The second condition states that the integral

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* A function u is of class $C^{(n)}$ in Ω when it is continuous within Ω and on the boundary, together with its derivatives of all orders less than or equal to n. If u is continuous throughout Ω, and if Ω may be divided into a finite number of sub-regions such that u is of class $C^{(n)}$ in each closed sub-region, then u is of class $D^{(n)}$ in Ω. This notation is due to Bolza.

† The derivatives $J'(\epsilon)$ and $J''(\epsilon)$ are continuous, under the assumption that f is of class C''.

$$\delta^2 J = \int\int_{\Omega} \{\zeta_x^2 f_{pp} + 2\zeta_x\zeta_y f_{pq} + \zeta_y^2 f_{qq} + \zeta^2 f_{zz} + 2\zeta\zeta_x f_{zp} + 2\zeta\zeta_y f_{zq}\} dxdy$$

must have the same sign, or vanish, for all allowed variations; the arguments of the derivatives of f are $x, y, \phi, \phi_x, \phi_y$. It has been shown * that if

$$\alpha^2 f_{pp} + 2\alpha\beta f_{pq} + \beta^2 f_{qq}$$

is a positive definite form $\delta^2 J$ is positive, if a negative definite form $\delta^2 J$ is negative. In point of fact it is a necessary condition for either a minimum or a maximum that this form be at least semi-definite. This will be proved without transforming the second variation, $\delta^2 J$.†

The theorem to be proved is then the following:

A necessary condition for an extremum is that

$$(1) \qquad f_{pp} f_{qq} - f_{pq}^2 \geqq 0$$

at all points of Ω.

Suppose, to the contrary, that the first member of (1) is negative at a point (x_0, y_0) of Ω. Then a region R containing the point (x_0, y_0) may be chosen, so small that throughout R ϕ_x and ϕ_y are continuous and

$$(2) \qquad f_{pp} f_{qq} - f_{pq}^2 < -m^2,$$

where m is a fixed number. There exist, therefore, two real and distinct roots of the equation

$$Q(h) \equiv f_{pp} h^2 + 2 f_{pq} h + f_{qq} = 0$$

for each point of R. $Q(h)$ will have different signs according as h lies between or outside these roots. Since the coefficients of $Q(h)$ are continuous within R, it follows that if R be chosen small enough, two fixed intervals P and N and a fixed number δ may be found, such that at all points of R

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* See Kneser, Encyklopädie der Mathematischen Wissenschaften, II A8, § 25.

† The necessity of Legendre's condition in the case of simple integrals could be shown in a similar manner, without transforming the second variation.

$$(3) \qquad \begin{aligned} f_{pp}h^2 + 2f_{pq}h + f_{qq} &> \delta^2 \textit{ for all } h \textit{ in } P, \\ f_{pp}h^2 + 2f_{pq}h + f_{qq} &< -\delta^2 \textit{ for all } h \textit{ in } N. \end{aligned}$$

Choose for the region R a rhombus, whose center is at (x_0, y_0) and whose sides are formed by the lines

$$(a) \quad u_1 - \epsilon = 0, \qquad (c) \quad u_1 + \epsilon = 0,$$

$$(b) \quad u_2 - \epsilon = 0, \qquad (d) \quad u_2 + \epsilon = 0,$$

where

$$u_1 = (x - x_0) \cos \alpha_1 + (y - y_0) \sin \alpha_1,$$

$$u_2 = (x - x_0) \cos \alpha_2 + (y - y_0) \sin \alpha_2,$$

the positive constant ϵ being chosen so small that (3) holds in R. The angles α_1, α_2 will be determined later. The diagonals of the rhombus divide it into four triangles $\alpha, \beta, \gamma, \delta$, containing the sides a, b, c, d, respectively. An allowed variation ζ is defined by the following equations, ζ being zero outside of R:

$$\zeta = -u_1 + \epsilon \quad \text{in } \alpha, \qquad \zeta = u_1 + \epsilon \quad \text{in } \gamma,$$

$$\zeta = -u_2 + \epsilon \quad \text{in } \beta, \qquad \zeta = u_2 + \epsilon \quad \text{in } \delta.$$

Now

$$\delta^2 J = \delta_1^2 J + \delta_2^2 J,$$

where

$$\delta_1^2 J = \int\int_R \zeta_y^2 \left\{ \left(\frac{\zeta_x}{\zeta_y} \right)^2 f_{pp} + 2 \frac{\zeta_x}{\zeta_y} f_{pq} + f_{qq} \right\} dxdy,$$

$$\delta_2^2 J = \int\int_R \{ \zeta^2 f_{zz} + 2\zeta\zeta_x f_{zp} + 2\zeta\zeta_y f_{zq} \} dxdy.$$

Since $|\zeta_x| \leqq | 1$, $|\zeta_y| \leqq 1$, $|\zeta| \leqq \epsilon$, it follows that

$$(4) \qquad |\delta_2^2 J| \leqq \epsilon M A (\epsilon + 4),$$

where M is the greatest of the maxima of $|f_{zz}|$, $|f_{zp}|$, $|f_{zq}|$, in R, and A is the area of R.

Now choose the angles α_1 and α_2 so that both of the quantities $\cot\alpha_1$, $\cot\alpha_2$ lie either in the interval P or in the interval N. Then the value of ζ_x/ζ_y will lie either in P or in N; if in P δ_1^2J will be positive, if in N δ_1^2J will be negative, on account of (3). In either case

$$|\delta_1^2J| > \delta^2\int\int_R \zeta_y^2 dxdy = \delta^2\frac{A}{2}(\sin^2\alpha_1 + \sin^2\alpha_2),$$

where A is the area of R. Therefore, from (4),

$$\delta^2J > A\left[\frac{\delta^2}{2}(\sin^2\alpha_1 + \sin^2\alpha_2) - \epsilon M(\epsilon + 4)\right]$$

if $\cot\alpha_1$ and $\cot\alpha_2$ lie in P, and

$$\delta^2J < A\left[-\frac{\delta^2}{2}(\sin^2\alpha_1 + \sin^2\alpha_2) + \epsilon M(\epsilon + 4)\right]$$

if $\cot\alpha_1$ and $\cot\alpha_2$ lie in N. If ϵ be taken sufficiently small, δ^2J may therefore be made positive or negative at pleasure. This result, obtained under the supposition that the inequality (1) is not satisfied at some point in Ω', contradicts the assumption that the function ϕ makes the integral J an extremum. *The condition* (1) *is therefore necessary for an extremum.*

The above proof of the necessity of condition (1) depends upon a variation whose first derivatives have finite discontinuities. If the extremum problem be so stated that the functions z are required to satisfy more restrictive conditions of continuity the proof must be altered, since the variations must satisfy these conditions also.

Suppose * the functions z are required to be of class C''. For the region R, chosen in the previous case as a rhombus, take the region formed by the sub-regions R_1, R_2, R_3, defined in the following manner: R_1 is a rectangle having the sides

$$u = \epsilon, \quad v = d, \quad u = -\epsilon, \quad v = -d,$$

where

$$u = (x - x_0)\cos\alpha + (y - y_0)\sin\alpha,$$

$$v = (x - x_0)\sin\alpha - (y - y_0)\cos\alpha.$$

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* This is the assumption usually made in dealing with double integrals.

The regions R_2 and R_3 are the portions exterior to R_1 of the areas bounded by the closed curves

$$u^2 + (v - d)^4 - \epsilon^2 = 0, \qquad u^2 + (v + d)^4 - \epsilon^2 = 0,$$

respectively.

A variation ζ of class C'' is defined by the following equations, ζ being zero outside of R:

$$\begin{aligned} \zeta &= (u^2 - \epsilon^2)^4 && \text{in } R_1, \\ \zeta &= (u^2 + (v - d)^4 - \epsilon^2)^4 && \text{in } R_2, \\ \zeta &= (u^2 + (v + d)^4 - \epsilon^2)^4 && \text{in } R_3. \end{aligned}$$

Now

$$\begin{aligned} \delta^2 J = \iint_{R_1} \zeta_y^2 \left\{ \left(\frac{\zeta_x}{\zeta_y}\right) f_{pp} + 2\frac{\zeta_x}{\zeta_y} f_{pq} + f_{qq} \right\} dxdy \\ + \iint_{R_1+R_2} \{ \zeta_x^2 f_{pp} + 2\zeta_x\zeta_y f_{pq} + \zeta_y^2 f_{qq} \} dxdy \\ + \iint_R \{ \zeta^2 f_{zz} + 2\zeta\zeta_x f_{zp} + 2\zeta\zeta_y f_{zq} \} dxdy. \end{aligned}$$

Since $|u| \leqq \epsilon$ in R, $|v - d| \leqq \sqrt{\epsilon}$ in R_2 and $|v + d| \leqq \sqrt{\epsilon}$ in R_3, it follows that at all points of R, $|\zeta_x| <$ *const.* ϵ^7, $|\zeta_y| <$ *const.* ϵ^7. Throughout R, $|\zeta| \leqq \epsilon^8$. The second and third integrands in the second member of the above equation are therefore numerically less than *const.* ϵ^{14} and *const.* ϵ^{15} respectively. The area of R is less than $4(d + \sqrt{\epsilon})\epsilon$, and the area of $R_2 + R_3$ is less than $4\epsilon^{\frac{3}{2}}$. Hence, for all values of ϵ sufficiently small,

$$\left| \delta^2 J - \iint_{R_1} \zeta_y^2 \left\{ \left(\frac{\zeta_x}{\zeta_y}\right)^2 f_{pp} + 2\frac{\zeta_x}{\zeta_y} f_{pq} + f_{qq} \right\} dxdy \right| < K.\epsilon^{\frac{31}{2}},$$

where K is a constant. Now in R_1

$$\zeta_x / \zeta_y = \cot \alpha.$$

Therefore, if $\cot \alpha$ lies in the interval P, the integral in the first member of the above inequality is positive; if $\cot \alpha$ lies in N, it is negative. In either case, on account of (3), its absolute value is greater than

$$\delta^2 \int\int_{R_1} \zeta_y^2 dxdy = 64\ \delta^2 \sin^2 \alpha \int_{-d}^{d}\int_{-\epsilon}^{\epsilon} (u^2 - \epsilon^2)^6 u^2 du dv = c^2\epsilon^{15},$$

where c is a constant independent of ϵ.

Therefore, if $\cot \alpha$ lies in the interval P,

$$\delta^2 J > \epsilon^{15}(c^2 - K\sqrt{\epsilon}).$$

If $\cot \alpha$ lies in the interval N,

$$\delta^2 J < \epsilon^{15}(-c^2 + K\sqrt{\epsilon}).$$

If ϵ be taken sufficiently small $\delta^2 J$ may therefore be made positive or negative at pleasure, and the necessity of condition (1) follows at once.

SHEFFIELD SCIENTIFIC SCHOOL,
YALE UNIVERSITY.

SHORTER NOTICES.

Handbuch der Theorie der Gammafunktion. By Dr. NIELS NIELSEN, Docent in Mathematics at the University of Copenhagen. Leipzig, Teubner, 1906. 326 pages.

As the name indicates, this book is a comprehensive handbook of the theory of the gamma function; it contains nearly all the known formulas, gives very exhaustive references, and arranges the entire material into a systematic and readable whole. Besides a full index, a bibliography of over five hundred titles is appended to the volume, with exact citation including date and number of pages of each memoir. That Dr. Nielsen is fitted for the task which he had set himself is attested by the long list of his own contributions, exceeded only by those of Euler, Lerch and Schlömilch.

The book is divided into three parts: the first treats of series and the analytic theory, the second of the theory of the definite integral, and the third of the inverse problem of expanding other series in terms of gamma functions.

The first part (112 pages, 581 numbered formulas) commences with the difference equation having $\Gamma(x)$ for a particular solution; Euler's constant is derived, and $\Gamma(x)$, $B(x, \frac{1}{2})$ defined so as to apply to irrational and complex values of the arguments. The functions $P_a(x)$, $Q_a(x)$ are then introduced and similarly discussed. The various functions are all developed into series, a chapter on the evaluation of infinite series and infinite products being added. The Stirling numbers and polynomials receive a whole chapter for their treatment. The functions previously defined are then expressed in terms of them.

A short chapter treats of zero points and of limiting values for infinite values of the argument. This part closes with Hölder's theorem that no algebraic differential equation can have $\Gamma(x)$ for a solution, but the proof is so given that the theorem applies at once to a number of related functions. Mention is occasionally made of gaps in the theory that have not been bridged, as, *e. g.*, the nature of Euler's constant, the roots of $\beta(x)$, the convergence of certain series, etc.

The second part (122 pages, 563 numbered formulas) begins with a very general consideration of functional equations which include the gamma function and the related functions as particular cases. The work is so general and so complete as to make it unsuitable for a textbook, though all the more valuable as a handbook. In the discussion of integration around a contour in the complex plane, the value of the book would have been enhanced and the volume made more easy of ready reference, had the path of integration been indicated by a figure. While the exact form is given in the text in each case, it is frequently necessary to hunt carefully for it; a reader using a formula for reference might easily mistake the path, understanding an ordinary rectilinear integration. The treatment of the asymptotic series, known as Stirling's series, is far too meagre, though no attempt is made at numerical evaluation anywhere in the book. The converging series involving Euler's constant is of little use for numerical purposes, while the asymptotic series can, by use of the functional equation, be used to determine the value of $\Gamma(x)$ to any number of places.

The third part (64 pages, 271 numbered formulas) is concerned with the inverse problem of expressing certain other functions in terms of $\Gamma(x)$ as function of development. This part is essentially the work of the author; its main accomplishment is the rigorous establishment of various results given by Stirling, and the pointing out that certain series derived formally by Gauss and others are not convergent within the region considered. A number of functions not expressible in convergent series have been defined asymptotically. Skillful manipulations of sums and products of series are given, and the possible region of continuation defined. Finally, these various facultative series are applied to the gamma function itself, and the various related functions.

The typographical work is excellent. A few misprints have crept in, some of which might lead to confusion when the book is used for reference. I mention the following:

Page 31, line 8 (from bottom), for 3 read 2, 3.
" 41, " 12 " " " Jefferey " Jeffery.
" 52, " 2 " " " $\psi\left(\frac{x+1}{2} + C\right)$ read $\psi\left(\frac{x+1}{2}\right) + C$.
" 58, equation (4), for q read s.
" 61, " (10), " $a_0x^{n'}$ " a_0x^{n}.
" 62, " (12), " (cotg " cotg (.
" 65, " (11), " $e^{+\frac{y}{x+s}}$ " $e^{-\frac{y}{x+s}}$.
" 103, line 3, insert Differential before Gleichung.
" 106, equation (7), for ϕ_n^{an} read $\phi_n^{a_n}$.
" 120, " (3), " $\psi(t)$ " $\phi(t)$.
" 129, " (5), " cs θ " cos θ.
" 131, " (2), " s " s.
" 136, " (17), insert dt.
" 144, line 1, for e^- read e^{-t}.
" 230, equation (3), for t read z.
" 231, " (8), " t " z.
" 233, " (13), insert dt and lower limit 0.
" 234, " (19), for $s = 0$ read 0.
" 257, " (13), " $x = 1$ " $x - 1$.
" 265, " (3), " $\infty \cdot \frac{1}{a}$ " ∞.
" 268, line 5, for füt read für.
" 279, equation (5), for ω read n.
" 282, " (7), insert f after first integral sign.
" 285, " (1), " -1 in brackets, $x - 1$ is exponent of t.
" 288, in the formula, insert dt after sign of integration.

VIRGIL SNYDER.

Mélanges de Géométrie à Quatre Dimensions. Par E. JOUFFRET. Paris, Gauthier-Villars, 1906. 227 pp.

IN the preface to this work the author expresses its purpose as follows: "Dans l'avant-propos de notre traité élémentaire de géométrie à quatre dimensions (page xviii), nous avons avancé que les théories classiques peuvent trouver d'utiles inspirations dans ce quatrième champ que nous avons appellé l'étendue, monde géométrique plus vaste que le leur, où des droites, des plans et des espaces sont déterminés respectivement par deux, trois, et quatre points, et sont coupés par un espace suivant des points, des droites, et des plans. Nous nous proposons de donner maintenant quelques applications de nature à justifier cette proposition."

The book contains eight chapters bearing the following titles: Coup d'œil sur les principes de la géometrie à quatre dimensions; Le système de coordonnées et les trois premiers polyèdres réguliers; L'hexagramme de Pascal; La surface du troisième degré; L'hexagramme et l'hexastigme; Les hypersurfaces du second degré; Les quartiques; La question de l'existence réelle de l'hyperespace. Few errata have been noted by the reviewer, and generally they are of such a nature as not to occasion any difficulty. A curious oversight occurs on page 8, line 4, where each of the equations $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0$ is spoken of as defining a plane. On the preceding page it is expressly stated that, in the geometry of four dimensions, a plane is defined by two linear relations among the variables. Each chapter closes with a brief summary giving in a concise form the results that have been obtained. In chapters involving many details these summaries are very useful. In Chapter VIII arguments are drawn from various sources in support of the existence of hyperspace. The author states that these are given because they are of interest, although the question does not enter in the field of mathematics. All that the author desires is simply to establish the utility of the idea.

One might be tempted to criticize the book on the ground of lack of unity. The author first devotes his attention to a discussion of the system of coordinates in the geometry of four dimensions; he then inserts two chapters, one dealing with a problem in the plane and the other dealing with a problem in space, before he comes back to the four dimensional geometry.

However, he has chosen a title for his book which permits considerable freedom both as to the matter presented and as to the order of presentation.

PETER FIELD.

Neuere Darstellungen der Grundprobleme der reinen Mathematik im Bereiche der Mittelschule. By ALOIS LANNER. Berlin, Otto Salle, 1907. viii + 192 pp. Price, 3 Marks.

ONE of the distinctive features of the teaching of elementary mathematics in the latter part of the nineteenth century was the influx of a large body of new theory relating to the fundamental laws underlying the common processes. Perhaps the best evidence of this is found in the lectures of Weierstrass, and among the best exponents of the movement is Stolz's "Allgemeine Arithmetik." It was natural to expect, however, that Weierstrass would never directly reach the teaching body of Germany, and that works as elaborate as those of Stolz would have but little practical influence in the schools.

The opening of the twentieth century is seeing an effort to bring the results of such labors as these to the attention of those who teach the elements. Naturally this involves a great deal of experiment. The college professor, with little knowledge of the powers and interests and immediate needs of preparatory students, is liable to insist upon secure foundations for every process, while the teacher in the classroom is equally likely to err the other way. For those who try to see the argument of each of these types, and to weigh them judicially, any effort to simplify the labors of the theorists and to present them in concise form, is very welcome.

Dr. Lanner has attempted exactly the work. He has not written a textbook, nor a work on the theory of teaching, but he has prepared a simple treatise that seeks to supplement each. In brief, it may be described as a handbook, giving in simple form the principles underlying each of the chapters of arithmetic and algebra as taken up in the elementary courses, and offering material for supplementing the theory of the textbook.

The general nature of the work can best be judged by a few of the chapter heads: Gleichheit und Grösse; Die natürlichen Zahlen, followed by the various operations; Erweiterung des Zahlengebietes durch die Subtraktion, introducing negative numbers; Die Teilbarkeit der ganzen Zahlen, both absolute and relative; Die systematische Darstellung der ganzen Zahlen, with the operations; Erweiterung des Zahlengebietes durch die Division, followed by the various forms of fractions; Das Rechnen mit endlichen Grenzwerten, with the relation between the decimal and common fraction; Irrationale Zahlen und Grenzwerte, unfortunately little more than a title; Das Rechnen mit algebraischen Funktionen; Die Veränderlichkeit der Funktionen, with a brief treatment of the differential coefficient and with a hint at maxima and minima; Die Integration, a mere introduction of four pages; Die Verhältnisse, followed by proportion; Die Teilregel, Mischungs- und Gesellschaftsrechnung, a subject that seems never to lose its fascination with the German arithmetician; Prozentrechnung, Rabatt, Diskonto, und Terminrechnung; Regeldetri und Kettensatz; Die Gleichung ersten Grades mit einer Unbekannten, followed by two unknowns; Rechnungsoperationen dritter Stufe, *i. e.*, powers, roots, and irrational equations; Die Logarithmen, with exponential equations; Die Ableitung von Funktionen aus ihren Eigenschaften; Imaginäre und komplexe Zahlen; Quadratische Gleichungen, followed by indeterminate equations, series, and interest computation; Kombinatorik, followed by the binomial theorem; Politische Arithmetik, including probabilities and insurance. Doubtless under the influence of that annoying parsimony which frequently characterizes the German publisher, the book has neither table of contents nor index, and hence it loses much of the value that it might possess as a handy tool.

From the list given above it will be seen that, while the theory of arithmetic is fairly covered, with the exception of the irrational number, the supplementary material in algebra is less satisfactory. Indeed, in spite of the fact that the author seeks to establish solid foundations for algebra, the result at once strikes the reader as incomplete. Possibly this is necessary, considering the purpose he had in mind, but it raises the question as to whether the work is worth the doing in this manner.

In his numerous historical notes Dr. Lanner is not altogether fortunate. Harriot, for instance, was not a professor at Oxford, nor did he die in 1622; it is not known that Recorde was born in 1510, nor that Paciuolo died in 1514; Newton was not born in 1643; Lagrange was not working in Turin in any of the years when he is said to have been there, and Jordanus Nemorarius probably did not die in Rome. The practice of placing a date after a man's name is helpful when it has any definite significance, but with Dr. Lanner it has none. Sometimes he gives the date of death, as of Stevin (1620); sometimes the date of a book, as in the case of Vieta (1591); but quite as often the date has no particular significance, as in the cases of Tartaglia (1557) and Riese (1550). Usually, however, the important date is omitted entirely, as with Stevin's work on decimals (1585). There are also several unfortunate errors in technical and proper names, as in the cases of Ouchtred, Boëthius and Boethius, Goss (for Coss), and Wimburgh (for Edinburgh).

Altogether, therefore, the work may be said to have been written with a laudable purpose, but to leave the field open for a carefully prepared treatise on the same subject.

DAVID EUGENE SMITH.

Reformvorschläge für den mathematischen und naturwissenschaftlichen Unterricht; entworfen von der Unterrichtskommission der Gesellschaft deutscher Naturforscher und Aerzte. Zweiter Teil: *Vorschläge überreicht der 78 Naturforscherversammlung in Stuttgart, 1906.* Herausgegeben von A. GUTZMER, Leipzig, Teubner, 1906. Pp. 73.

THE work of the commission appointed by the German society of natural scientists and physicians, as summarized in this second annual report, has been carried on during the past year along the lines laid down in its first report which has already been reviewed in the BULLETIN * So far as mathematics is concerned, the present report consists of a discussion of the way in which the principles laid down in the first report would work out in practice in certain special types of German schools not there considered, and brings little of general significance beyond the boundaries of Germany.

The section on instruction in mathematics in the higher schools for girls is of some interest to us in America, where thousands of girls and young women annually carry through successfully work of a grade as advanced and as difficult as any that is given to the young men in the German gymnasia. The commission urges that more work in mathematics be given in the schools for girls, and, indeed, that those who complete the full course should end with substantially the same attainments as the young men who have passed through a gymnasium. In the earlier years (to age fifteen or sixteen) the commission recommends somewhat more emphasis upon intuition and practical exercises, especially the making of models, than in corresponding work in the boys' schools, where the logical element must be brought more freely into play. The avoidance of pedantic proofs of statements that are intuitionally evident is particularly important in teaching girls, special emphasis should be laid on the esthetic side of mathematics, while the logical side should not be neglected; the instruction as a whole must have its own peculiar character, and be based upon textbooks specially prepared therefor.

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* Vol. 12, (1906), pp. 347-351.

In the later years of the course no special differences in the instruction of young men and women are recommended, since the very fact that the women go on into the later years is of itself sufficient evidence to the commission of a somewhat mannish type of mind that may reasonably be expected to prove equal to the logical demands made on it by the mathematics to be given, extending perhaps as far as the elements of analytic geometry.

J. W. A. YOUNG.

N.-H. Abel, sa Vie et son Oeuvre. Par CH. LUCAS DE PESLOÜAN. Paris, Gauthier-Villars, 1906. Pp. xiii + 168.

No pretension is made in this biography to add to the facts previously published relating to Abel. It will not supersede the biography written by C.-A. Bjerknes and brought out in 1885 by the same publishers. The aim of De Pesloüan is to put the story of the short life of the gifted Norwegian in attractive form for scientific readers who are not specialists along the lines of Abel's researches. The idea is a commendable one and should be carried out more frequently in biographies of scientists. The author exhibits great sympathy and admiration for Abel. In the preface he goes so far as to call Abel's researches "l' oeuvre du plus grand mathématicien du XIX^e siècle." More judicious and suggestive is a remark which is attributed by the author to Hermite (page 136), by some others to Sylvester: "Abel a laissé aux mathématiciens de quoi travailler pendant cent cinquante ans."

It is well known that Abel, in his twenty-third year, started upon a journey to continue his studies and meet the mathematicians of his day, that this diffident and sensitive youth was cordially received by Crelle in Berlin, that he went to Paris without stopping on the way to see Gauss at Göttingen, who he imagined had slighted him by paying no attention to his first draft of the proof of the impossibility of an algebraic solution of the quintic, and that in Paris he failed to receive appreciation and stimulus from Cauchy, Dirichlet, Poisson, Legendre, Fourier, Arago, Ampère and others. An article which Abel submitted to the French academy, containing what is known now as "Abel's theorem" on "abelian functions," was lost sight of, and not published until twelve years after his death. De Pesloüan discusses at length the causes of this lack of recognition. He holds (page 130) that at this period French mathematicians had quite generally abandoned the cultivation of pure mathematics, to take up applications to heat, elasticity and electricity, that the failure to appreciate the importance of the paper was not, as charged in a sketch of Abel in Michaud's Biographical Dictionary, due to indifference, nor to egoism. "Si les savants furent coupables envers lui, ce fut par suite de leur ignorance ou plutôt de leur incompréhension."

When Abel was in Paris, Galois was sixteen years old and had not yet made his début in mathematics. Had the lives of these two men of genius been spared longer to science, what further revelations might they not have made!

De Pesloüan says, page 109, that Abel's second paper on "Recherches sur les fonctions elliptiques" is lost. He does not seem to be aware that it was found by G. Mittag-Leffler in 1894 and published in the *Acta Mathematica* (volume 26, pages 1–42).*

Some errors in the spelling of proper names and in the statement of theorems might have been avoided by greater care in the reading of the proofs.

FLORIAN CAJORI.

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* G. Eneström in *Bibliotheca Mathematica*, ser. 3, vol. 7 (1906), page 217.

NOTES.

At the meeting of the American Mathematical Society on April 27, Professor W. F. Osgood will deliver his presidential address, postponed from the recent annual meeting. The subject of the address will be "The calculus in colleges and technical schools."

The Annual Register of the American Mathematical Society for 1907, containing the list of officers and members of the Society, constitution and by-laws, annual reports, and catalogue of journals in the Society's library together with other accessions for 1905–1906, has recently been issued. A complete catalogue of the library up to 1905 is contained in the Register for that year, copies of which may still be obtained from the Secretary.

The opening (January) number of volume 8 of the *Transactions* of the American Mathematical Society contains the following papers: "Generalization of the groups of genus zero," by G. A. Miller; "On reflexive geometry," by F. Morley; "The groups in which every subgroup is either abelian or hamiltonian," by G. A. Miller; "On modular groups isomorphic with a given linear group," by H. F. Blichfeldt; "Denumerants of double differentiants," by W. E. Story; "The groups of classes of congruent matrices, with application to the group of isomorphisms of any abelian group," by A. Ranum; "A theorem of Abel and its application to the development of a function in terms of Bessel's functions," by C. E. Smith; "Irreducible linear homogeneous groups whose orders are powers of a prime," by W. B. Fite; "Applicable surfaces with asymptotic lines of one surface corresponding to a conjugate system of the other," by L. P. Eisenhart.

At the meeting of the London mathematical society held on January 10 the following papers were read: By A. Stott, "Exhibition of four-dimensional models"; by E. W. Hobson, "On the uniform convergence of Fourier series"; by J. E. Littlewood, "Asymptotic approximation of integral functions of zero order"; by A. R. Forsyth, "Partial differential equations of the second order having integral systems free from partial quadratures"; by G. H. HARDY, "On the singular points of some classes of power series in several variables"; by W. BURNSIDE, "The construction of the line drawn through a given point to meet two given lines"; by P. W. WOOD, "On the reduction of covariants of binary quantics of infinite order, part III"; by A. CUNNINGHAM, "On hypereven numbers, and on Fermat's numbers."

THE annual meeting of the British mathematical association was held at King's College, London, January 26. Professor G. H. BRYAN was elected president for the ensuing year. The following papers were read: by H. H. HUDSON, "Diagrams of anemoids"; by A. LODGE, "On contracted methods in arithmetic"; by C. S. JACKSON, "On the elementary arithmetic of the theory of numbers." The association now includes 419 members.

AT the annual public meeting of the Paris academy of sciences, held on December 17, 1906, the following prizes were awarded for mathematical memoirs: Grand prize (fr. 3000) divided among H. PADÉ (fr. 1500), R. DE MONTESSUS (fr. 1000) and M. AURIC (fr. 500); Francoeur prize (fr. 1000) to E. LEMOINE; Poncelet prize (fr. 2000) to L. GUICHARD; Montyon prize (fr. 700) to G. MARIÉ; Lalande prize (fr. 540) divided between R. G. AITKEN and W. J. HUSSEY; Valz prize (fr. 460) to J. PALISA; Janssen medal to A. RICCÒ. The Wilde prize (fr. 4000) for general progress in science was awarded in part to M. MASSAU for his work on graphic integration; the Laplace prize (complete works of Laplace) to M. LÉVY. The subjects for which the prizes were offered were announced in the BULLETIN, volume 11, page 327, and volume 12, page 319.

The Academy announces the following prizes in mathematical sciences for 1908 and 1909; competing memoirs should be in the hands of the secretary before December 31 of the year preceding that in which the respective prizes are awarded:—The Grand prize and Francoeur prize as announced before, the Poncelet prize in 1908 for general progress in pure mathematics, in 1909 for general progress in applied mathematics; the Bordin prize (fr. 3000) for the following problem:

"The absolute invariant which represents the number of double integrals of the second kind on an algebraic surface depends upon a relative invariant ρ, which plays an important part in the theory of integrals of total differentials of the third kind and in the theory of algebraic curves traced upon the surface. It is proposed to make an exhaustive study of this invariant and to determine in particular how its exact value can be found, at least for an extended category of surfaces" (competing memoirs must be received before January 1, 1909).

The Montyon, Fourneyron, Vaillant and Borleau prizes are to be awarded under the conditions previously announced.

COLUMBIA UNIVERSITY.—The following special courses of public lectures in mathematical physics are announced:—By Professor OTTO LUMMER, of the University of Breslau: The laws of radiation of the black body; their theoretical establishment and practical importance. Ten lectures, Fridays at 4 P. M., Saturdays at 10 A. M., beginning Friday, February 15.—By Professor JOSEPH LARMOR, of Cambridge University: Physical dynamics. Six lectures, Wednesday–Thursday, March 27–28; Friday–Saturday, April 5–6, April 12–13. The Wednesday and Friday lectures are given at 4 P. M., the Thursday and Saturday lectures at 10 A. M.

Detailed information regarding these courses is contained in the circulars issued by the University.

DR. R. BAIRE, associate in mathematics at the University of Dijon, has been promoted to a full professorship of mathematics at the same institution.

THE British royal astronomical society has awarded its gold medal for 1907 to Professor E. W. BROWN for his work on the motion of the moon.

PROFESSOR R. MÜLLER, of the technical school at Brunswick, has been appointed professor of mathematics at the technical school of Darmstadt.

PROFESSOR C. K. RUSSJAN, of the technical school at Lemberg, has been appointed professor of pure mathematics at the University of Charkow.

PROFESSOR K. BOEHM, of the University of Heidelberg, has been promoted to a full professorship, with special regard to elementary mathematics.

DR. E. W. HOBSON has been appointed chairman of the mathematical tripos, part II, for 1907.

PROFESSOR C. A. NOBLE, of the University of California, has been granted leave of absence and will spend next year in study at European universities.

DR. C. C. JONES has been appointed chancellor and professor of mathematics at the University of New Brunswick, to succeed the late Professor Harrison.

PROFESSOR G. W. JONES, of Cornell University, will retire from active service at the close of the present academic year with the title of professor emeritus.

AT the University of Texas, Professor H. Y. BENEDICT has been appointed professor of applied mathematics.

MR. G. A. ROSS, formerly of Hardin College, has been appointed instructor in mathematics at the George Washington University.

PROFESSOR E. JURGENS, of the technical school at Aachen, died January 5, 1907, at the age of 56 years. Professor A. MANNHEIM, professor emeritus at the Ecole Polytechnique of Paris, died December 11, 1906, at the age of 75 years. Professor J. A. OUDEMANS, of the University of Utrecht, died December 13, 1906, at the age of 78 years.

CATALOGUES of second-hand books: Martin Boas, Berlin, Karlstrasse 25, catalogue No. 60, 3082 titles in mathematics, physics and astronomy. — W. Heffer and Sons, Cambridge, Eng., catalogue No. 22, 1029 titles in mathematics and science, including the library of the late Professor Joly. — W. Junk, Berlin, Kurfürstendamm 201, 1629 titles in mathematics and science.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ARNOUX (G.). Arithmétique graphique Introduction à l'étude des fonctions arithmétiques. Paris, Gauthier-Villars, 1906. 8vo. 20 + 225 pp. Fr. 5.00

AUZEMBERGER. See VESSIOT (E.).

BERNSTEIN (R.). Les courbes du troisième degré. (Thèse.) Besançon, 1906.

BERTRAND (J.). Calcul des probabilités. 2e édition conforme à la 1re. Paris, Gauthier-Villars, 1907. 8vo. 57 + 322 pp. Fr. 12.00

Carver (W. B.). On the Cayley-Veronese class of configurations. (Diss., Johns Hopkins.) Lancaster, New Era Printing Co., 1906. 8vo. 22 pp. and pp. 534–545.

Fiske (T. S.). Functions of a complex variable. New York, Wiley, 1906. 8vo. (Mathematical monograph series.) Cloth. $1.00

Maltbie (W. H.). Analytic geometry: a first course. Baltimore, The Woman's College, 1906. 3 + 142 pp. $0.25

Papelier (G.). Précis de géométrie analytique, à l'usage des élèves de mathématiques spéciales, conforme au programme du 26 juillet 1904. Paris, Vuibert, 1907. 8vo. 304 pp.

Riollot (J.). Les carrés magiques. Contribution à leur étude. Paris, Gauthier-Villars, 1907. 8vo. 124 pp. Fr. 5.00

Santerre (S.). Psychologie du nombre et des opérations élémentaires de l'arithmétique. La genèse des premières notions de l'arithmétique; notions de suite, de nombre, de somme, et de différence; produit et quotient. Avec une préface par P. Janet. Paris, Doin, 1907. 8vo. 14 + 182 pp.

Stuyvaert (M.). Les nombres positifs. Exposé des théories modernes de l'arithmétique élémentaire. Gand, van Goethem, 1906. 8vo. 133 pp. Fr. 3.00

Tafelmacher (A.). Elementos de la geometría analítica del plano, para el uso del curso militar de la Escuela militar. Santiago de Chile, 1906. 8vo. 135 pp. P. 1.00

Tannenberg (W. de). Cours de géométrie analytique. Paris, Vuibert et Nony, 1906. 4to. 2 fascicules: pp. 1–88

Vessiot (E.). Leçons de géométrie supérieure professées à l'Université de Lyon. Rédigées par Auzemberger. Paris, Hermann, 1906. 4to. 326 pp. (Autogr.) Fr. 12.00

Whitehead (A. N.). Axioms of projective geometry. Cambridge University Press, 1906. 8vo. 2s. 6d.

II. ELEMENTARY MATHEMATICS.

Amaldi (U.). See Enriques (F.).

Ballerini (P.). La geometria pel ginnasio superiore e pel liceo. Monza, Artigianelli, 1906. 16mo. 246 pp. L. 2.00

Barbisch (H.). See Jahnke (J.).

Baudoin (P.). See Bourlet (C.).

Bergmeister (H.). Geometrische Formenlehre fur Mädchen-Lyzeen. Teil 2: Für die III. und IV. Klasse. 2te, umgearbeitete Auflage. Wien, Deuticke, 1906. 8vo. 4 + 69 pp. Boards. M. 1.40

Böttcher (R.) und Sendler (R.). Raumlehre fur Lehrerseminare. Nach dem Lehrplan vom 1. VII. 1901 bearbeitet. Teil 1: Planimetrie. 2te Auflage. Breslau, Handel, 1907. 8vo. 4 + 106 pp. Boards. M. 1.25

Bourlet (C.) et Baudoin (P.). Cours abrégé de géométrie publié avec de nombreux exercices théoriques et pratiques et des applications au dessin géométrique. Paris, Hachette, 1906. 16mo. 8 + 408 pp. F. 2.50

Dozal y Obejero (A.). Nociones de aritmética y geometría. Madrid, Arias, 1906. 12mo. 235 pp. P. 10.00

CAVEZZALI (A.). L'aritmetica e la geometria ad uso delle scuole elementari, con numerosi esercizi orali e scritti. Per la III classe. Milano, Abbiati, 1907. 16mo. 70 pp. L. 0.25

CIAMBERLINI (C.). Nozioni di aritmetica e geometria, con esercizi di calcolo mentale e scritto e problemi per la quarta classe elementare. Firenze, Bemporad, 1906. 16mo. 132 pp. (Biblioteca scolastica.) L. 0.50

COMBETTE (E.). Cours de trigonométrie. 4e édition, entièrement refondue. Paris, 1906. 8vo. Fr. 4.00

COSTANZO (G.) e NEGRO (C.). Geometria intuitiva e rudimenti di disegno geometrico per le prime tre classi del ginnasio. 2a edizione riveduta e corretta. Bologna, Zanichelli, 1907. 16mo. 168 pp.

DHYVERS. Questions et réponses sur les mathématiques et la physique. (Programmes du 27 juillet 1905.) Baccalauréat. Première partie (sections A et B). Paris, Croville-Morant [1906]. 16mo. 63 pp. Fr. 1.25

DREYFUS (L.). See FORT (L.).

DUPORT (J. B.). Lehrbuch der Arithmetik für die zweite und dritte Klasse der Mädchen-Lyzeen. 2te Auflage. Wien, Deuticke, 1906. 8vo. Zweite Klasse. 3 + 83 pp. Boards. M. 1.20

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ENRIQUES (F.) e AMALDI (U.). Elementi di geometria. 2a edizione ridotta. Bologna, Zanichelli, 1907. 16mo. 279 pp. L. 2.50

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L R RY

THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and thirty-second regular meeting of the Society was held in New York City on Saturday, February 23, 1907. The attendance at the two sessions included the following thirty-three members:

Professor G. A. Bliss, Professor E. W. Brown, Dr. W. H. Bussey, Professor F. N. Cole, Dr. W. S. Dennett, Dr. G. B. Germann, Dr. A. M. Hiltebeitel, Professor Edward Kasner, Dr. G. H. Ling, Professor E. O. Lovett, Professor Max Mason, Mr. A. R. Maxson, Professor H. B. Mitchell, Dr. R. L. Moore, Mr. H. W. Reddick, Professor L. W. Reid, Dr. R. G. D. Richardson, Miss I. M. Schottenfels, Mr. L. P. Sicelof, Dr. Clara E. Smith, Professor P. F. Smith, Dr. R. P. Stephens, Dr. W. M. Strong, Professor J. H. Tanner, Professor H. D. Thompson, Dr. A. L. Underhill, Professor Oswald Veblen, Professor L. A. Wait, Mr. H. E. Webb, Professor H. S. White, Professor E. B. Wilson, Dr. Ruth G. Wood, Professor J. W. Young.

The President of the Society, Professor H. S. White, and Vice-President Professor P. F. Smith occupied the chair. The Council announced the election of the following persons to membership in the Society: Professor T. M. Focke, Case School of Applied Science, Cleveland, Ohio; Dr. D. C. Gillespie, Cornell University; Professor C. C. Grove, Hamilton College, Clinton, N. Y.; Professor T. W. Palmer, University of Alabama; Professor N. A. Pattillo, Randolph-Macon Woman's College; Mr. F. D. Posey, University of Chicago; Miss Gertrude Smith, Vassar College; Dr. A. L. Underhill, Princeton University. Ten applications for admission to the Society were received.

After seven years' service as Treasurer, Dr. W. S. Dennett expressed the wish to be relieved of the duties of that office. The vacancy was filled by the appointment of Professor J. H. Tanner. Professor Maxime Bôcher was elected a member of the Editorial Committee of the *Transactions*, to succeed Professor E. H. Moore, who will retire, at the completion of the present volume of that journal, after eight years' service as editor in chief. Appropriate resolutions expressing grateful appreciation of the services of these two officers were adopted by the Society.

A standing committee of the Council, consisting of the Treasurer as chairman, Professor Brown, and Dr. Dennett, was appointed to have charge of the investment of the Society's life membership and surplus funds.

For the better regulation of the presentation of papers at the meetings of the Society, the rule was adopted that hereafter papers shall be read in the order and at the session assigned them on the printed programme. Papers whose reading is postponed at the request of the authors or in their absence will be read at the close of the session, if time permits, or at the close of the last session. Papers not announced on the printed programme, but subsequently accepted for presentation, may be read so far as time permits at the close of any session, after the printed list has been exhausted, or at the close of the last session.

To facilitate discussion, abstracts of papers will be included in the printed programme if furnished by the authors for that purpose at least three weeks in advance of the meeting. Such abstracts should be in the usual form, but should also be accompanied by standard and special references to the literature of the subject.

The following papers were read at this meeting:

(1) Professor R. D. Carmichael: "On dividing an angle into parts having the ratio of any given straight lines."

(2) Professor R. D. Carmichael: "A table of multiply perfect numbers."

(3) Professor G. A. Miller: "The groups generated by three operators each of which is the product of the other two."

(4) Dr. R. P. Stephens: "On a quintic with three parallel tangents in any direction" (preliminary communication).

(5) Professor E. B. Wilson: "On the revolutions of a dark body about the sun."

(6) Mr. C. N. Moore: "On the introduction of convergence factors into summable series and summable integrals."

(7) Professor G. A. Bliss: "The construction of a field of extremals about a given point."

(8) Dr. R. G. D. Richardson: "Differentiation and integration of definite integrals."

(9) Professor E. R. Hedrick: "On a final form of the theorem of uniform continuity."

(10) Professor R. D. Carmichael: "On the classification of quartic curves possessing fourfold symmetry with respect to a point."

Mr. Moore's paper was communicated to the Society through Professor Bôcher. In the absence of the authors the papers of Professor Carmichael, Professor Miller, Mr. Moore, and Professor Hedrick were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In this paper Professor Carmichael considers the problem of dividing an angle into parts having the ratios of any given straight lines, and effects a solution by means of the locus of the polar equation $\rho \sin \theta = \theta$. A method of constructing the curve by continuous motion is also given. Once the curve is constructed, the division of the angle into the proper parts is a very simple matter.

2. In his paper on multiply perfect numbers Professor Carmichael exhibits a method for determining all the multiply perfect numbers up to 1,000,000,000. He gives a table of these numbers. In addition he gives another table which contains such other numbers as are known to him to be multiply perfect. It is interesting to note that this computation does not discover a multiply perfect odd number. It therefore still remains an unsettled question whether multiply perfect odd numbers exist.

3. Professor Miller proved that if three operators are such that the product of any two is equal to the third they generate either a dihedral group or the quaternion group. When the three defining equations do not admit the cyclic permutation of the operators, the latter generate a dihedral group, and every dihedral group can be generated by three such operators. When the equations admit such a cyclic permutation, the three operators in question generate one of the following four groups: identity, the group of order two, the four-group, or the quaternion group. Moreover, the operators may be so chosen as to satisfy the given conditions and generate any one of these four groups.

4. In an article, "On the pentadeltoid" in the *Transactions* for April, 1906, Dr. Stephens discussed a quintic whose parametric equation is

$$t^5 + a\mu t^4 + xt^3 + ayt^2 + \mu t + a = 0,$$

to which can be drawn in any given direction only one tangent. A quintic of similar mechanical construction is

$$t^5 - xt^4 + \mu t^3 - a\mu t^2 + ayt - a = 0,$$

which is a curve of class five and degree eight. To this curve, say K, in any direction may be drawn three parallel tangents, the centroid C of whose points of tangency is independent of the given direction and is defined as the center of K. In general there are five cusps, but these may decrease by twos to one while the curve changes in shape from the five-cusped hypocycloid as one limit to the cardioid as the other. Inscribed in each curve K is a concentric ellipse which touches K in five points. The orthoptic curve is a limaçon which touches K in five points.

Corresponding theorems may be proved for the general case where the equation is

$$t^n + xt^{n-1} + a_1t^{n-2} + a_2t^{n-3} + \cdots + b_2t^3 + b_1t^2 + yt + 1 = 0;$$

for instance, this is a curve to which $n - 2$ parallel tangents may be drawn in any given direction.

5. In this paper Professor Wilson takes up mathematically the problem discussed by Poynting, in a lecture before the Royal Society recently printed in *Nature*. The differential equations of motion of a particle under the action of the sun's radiation as well as gravitational attraction are such that the equation of the orbit may readily be expressed in terms of Bessel functions. The discussion of the formulas brings out simple approximate expressions for the decrease of the radius vector, the diminution of the eccentricity, and the rotation of the line of apsides of the orbit. It also appears that the total number of revolutions which a particle can make about the sun in an infinite time is finite even when the particle starts at an infinite distance.

6. The most important results of Mr. Moore's paper are contained in the following two theorems:

I. If the series $u_0 + u_1 + u_2 + \cdots$ is summable in the sense of Frobenius (*Crelle*, volume 89) and has the value S, then the series $u_0 + u_1f_1(a) + u_2f_2(a) + \cdots$ will be absolutely convergent for all positive values of a, and its value $F(a)$ will be continuous for such values and will approach S as its limit when $\alpha = +0$, provided the convergence factors $f_n(\alpha)$ satisfy for all positive, integral values of n the following conditions: (a) $f_n(\alpha)$ is continuous $(\alpha \geqq 0)$; (b) $|f_n(\alpha)| < N / n^{2+\rho}\alpha^{2+\rho}$ $(\alpha > 0)$; (c) $f_n(0) = 1$; (d) $f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha) \geqq 0$ $(0 \leqq n\alpha \leqq c)$; ($e$) $|f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)| < L / n^{2+\rho}\alpha^{\rho}$ $(\alpha > 0)$; where N, ρ, c and L are positive constants.

II. If the function $f(x)$ is uniformly continuous for all values of $x \geqq a > 0$, and if the integral

$$\int_a^\infty f(x)dx$$

is summable and has the value S, that is, if

$$\lim_{x=\infty}\left(\frac{1}{x}\int_a^x \int_a^\alpha f(\beta)d\beta d\alpha\right) = S,$$

then the integral

$$\int_a^\infty f(x)\phi(\alpha, x)dx$$

will be absolutely convergent for all positive values of α, and its value $F(\alpha)$ will be continuous for such values and will approach S as its limit when $\alpha = +0$, provided the convergence factor $\phi(\alpha, x)$ satisfies the following conditions: (a) $\phi(\alpha, x)$ is continuous $(x \geqq a, \alpha \geqq 0)$; ($b$) $\phi_x''(\alpha, x)$ exists and is continuous $(x \geqq a, \alpha \geqq 0)$; ($c$) $|\phi(\alpha, x)| < N / x^{2+\rho}\alpha^{2+\rho}$ $(x \geqq a, \alpha > 0)$; (d) $\phi(0, x) = 1$ $(x \geqq a)$; (e) $\phi_x''(\alpha, x) \geqq 0$ $(0 \leqq x\alpha \leqq c)$; ($f$) $|\phi_x''(\alpha, x)| < L / x^{2+\rho}\alpha^{\rho}$ $(x \geqq a, \alpha > 0)$; where N, ρ, c and L are positive constants.

Other theorems are proved for the case in which the original series or integral is convergent.

7. Professor Bliss's paper appears in full in the present number of the BULLETIN.

8. On the basis of the results obtained in former papers for the equality of the double integral and the iterated integral, Dr. Richardson is able to derive general theorems for the inversion of the order of integration. If one of the integrals

$$\int_X dx \int_Y |f|\, dy, \quad \int_Y dy \int_X |f|\, dx$$

is finite, then

$$\int_X dx \int_Y f\, dy = \int_Y dy \int_X f\, dx.$$

This is true if certain restrictions are put on the arrangement of the points of discontinuity, these restrictions being necessary to insure integrability. These results are applied to a discussion of the continuity and differentiation of a definite integral depending on a parameter.

9. Professor Hedrick presented a final generalization of the theorem that a function continuous in a closed interval is uniformly continuous in that interval. The generalization is stated in terms of the oscillation $\omega(x, \delta)$ in an interval $(x - \delta, x + \delta)$ and the oscillation $\Omega(x)$ at a point. If then $\Omega(x) \leqq \lambda$ for values of x in a closed assemblage (E) for the values of $f(x)$ which correspond to values of x in any assemblage (H), then $\omega(x, \delta) < \lambda + \epsilon$ whenever $\delta < \eta$, for all x in (E), the order of choice being ϵ, η, x.

10. Professor Carmichael determines the simplest general forms of the equation of quartic curves having fourfold symmetry with respect to a point, the point being taken as the origin. He shows that there are the following five classes which may be combined into two groups, the first containing I, IV, and V, and the second II and III:

I. $x^4 + y^4 + c_1xy(x^2 - y^2) + c_2x^2y^2 + c_3(x^2 + y^2) + c_4 = 0.$

II. $x^4 - y^4 + r_1xy(x^2 + y^2) + r_2(x^2 - y^2) + r_3xy = 0.$

III. $x^3y + xy^3 + p_1(x^2 - y^2) + p_2xy = 0.$

IV. $x^3y - xy^3 + q_1x^2y^2 + q_2(x^2 + y^2) + q_3 = 0.$

V. $x^2y^2 + s_1(x^2 + y^2) + s_2 = 0.$

F. N. COLE,
Secretary.

THE CONSTRUCTION OF A FIELD OF EXTREMALS ABOUT A GIVEN POINT.

BY PROFESSOR G. A. BLISS.

(Read before the American Mathematical Society, February 23, 1907.)

IN establishing the sufficiency of the well-known conditions for the simplest problem of the calculus of variations in the plane, an essential step is the proof that a one-parameter family of extremals simply cover a portion of the plane and constitutes a so-called field. It has been shown by several writers * that when all the extremals of the family pass through a single point O, any arc of one of them can be imbedded in a field provided that it does not contain the point O or its conjugate. But it is also true that the field extends up to and surrounds O, as will be proved in the following paragraphs.†

When the point O is not a singular point of the problem it may, without loss of generality, be taken as the origin, and the equations of the extremals through it can always be found in the form

$$x = \phi(s, t), \quad y = \psi(s, t). \tag{1}$$

The parameter s is the length of arc measured from O, and t is the angle between the x-axis and the direction of the extremal at O. The functions ϕ and ψ therefore satisfy the identities

$$\begin{aligned} \phi_s^2 + \psi_s^2 \equiv 1, \quad 0 \equiv \phi(0, t), \quad 0 \equiv \psi(0, t), \\ \cos t \equiv \phi_s(0, t), \quad \sin t \equiv \psi_s(0, t), \end{aligned} \tag{2}$$

where the subscripts denote partial differentiation. They are furthermore supposed to be of class C'‡ for

$$0 \leqq s \leqq S, \quad t = \text{any value},$$

and, since one and only one extremal passes through O in a given direction, to be periodic with period 2π in t.

It is proposed to show that a circle of radius ρ can be found

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* Osgood, *Annals of Mathematics*, 2d ser., vol. 2 (1901), p. 112. Bolza, Lectures on the calculus of variations (1904), pp. 78, 175. Goursat, Cours d'analyse, vol. 2, p. 613.

† In his lectures on the calculus of variations, summer semester, 1879, Weierstrass stated that a field including the point O could be constructed, but omitted the proof.

‡ *I. e.*, continuous with continuous first partial derivatives.

with center at O such that the equations

$$(3) \qquad r\cos\alpha = \phi(s, t), \quad r\sin\alpha = \psi(s, t)$$

have one and only one solution (s, t) corresponding to any pair of values (r, α) for which

$$(4) \qquad 0 < r < \rho, \quad \alpha = \text{any value},$$

and which correspond therefore to a point in the circle. The pair of functions

$$(5) \qquad s = s(r, \alpha), \quad t = t(r, \alpha)$$

so defined turn out to be of class C' in the region (4) and periodic in α with period 2π.

First consider r as a function of s and t,

$$r^2 = \phi^2 + \psi^2.$$

The value of each of the expressions

$$(6) \qquad \frac{dr}{ds}, \quad \frac{\phi_s\psi_t - \phi_t\psi_s}{\phi\phi_s + \psi\psi_s}$$

at the point O $(s = 0)$ is unity. The derivative dr/ds has the value

$$(7) \qquad \frac{dr}{ds} = \frac{\phi\phi_s + \psi\psi_s}{\sqrt{\phi^2 + \psi^2}},$$

and by Taylor's formula

$$\phi(s, t) = s \cdot \phi_s(\theta s, t), \quad \psi(s, t) = s\psi_s(\theta' s, t).$$

By substituting these values in (7) and letting s approach zero the desired result for dr/ds is derived. If $\Delta(s, t)$ denotes the numerator of the second of the expressions (6), the value of Δ_s is

$$\Delta_s(s, t) = \phi_s\psi_{ts} - \phi_{ts}\psi_s + \phi_{ss}\psi_t - \phi_t\psi_{ss}.$$

At the point 0

$$\Delta_s(0, t) = 1$$

on account of the equations (2), from which may be derived also

$$\phi_{ts}(0, t) = -\sin t, \quad \psi_{ts}(0, t) = \cos t.$$

It follows then that the fraction

$$\frac{\phi_s\psi_t - \phi_t\psi_s}{\phi\phi_s + \psi\psi_s} = \frac{s\Delta_s(\theta''s,\ t)}{s\phi_s(\theta s,\ t)\ \phi_s + s\psi_s(\theta' s,\ t)\ \psi_s}$$

takes on the value unity when s approaches zero.

Since the expressions (6) are equal to unity at the point O ($s = 0$), there will exist a largest value S_1 such that the inequalities

$$(8) \qquad \frac{dr}{ds} > 0, \quad 2 > \frac{\phi_s\psi_t - \phi_t\psi_s}{\phi\phi_s + \psi\psi_s} > 0$$

hold for all points $(s,\ t)$ in the region

$$(9) \qquad 0 < s < S_1(\leqq S), \quad t = \text{any value}.$$

If ρ denotes the minimum value of $r(S_1,\ t)$, the circle of radius ρ about the point O will be simply covered by the extremals (1) and will be a circle, therefore, of the kind originally sought.

To prove this, consider any circle about O of radius $r_1 < \rho$. It is cut once and only once by each of the extremals (1), and consequently the equation

$$r_1^2 = \phi^2 + \psi^2$$

has one solution s in the interval (9) corresponding to each value of t. By the well-known theorems on implicit functions the function $s(t)$ so defined is of class C', since the derivative $\phi\phi_s + \psi\psi_s$ can not vanish (see (7) and (8)). When $r = r_1$ and this function $s(t)$ are substituted in the equations (3), α is determined as a function of t whose derivative by an easy calculation is found to be

$$\frac{d\alpha}{dt} = \frac{\phi_s\psi_t - \phi_t\psi_s}{\phi\phi_s + \psi\psi_s}.$$

From (8) it follows therefore that α always increases as t increases from 0 to 2π. Furthermore the value $\alpha(2\pi)$ differs from $\alpha(0)$ by some multiple of 2π, since there is only one extremal through the point 0 in the direction $t = 0$, or what is the same thing, in the direction $t = 2\pi$. But from (8) again

$$\alpha(2\pi) < \alpha(0) + 2 \cdot 2\pi,$$

and one sees therefore that $\alpha(2\pi) = \alpha(0) + 2\pi$. It is evident that the two intervals $\alpha(0) \leqq \alpha \leqq \alpha(0) + 2\pi$ and $0 \leqq t \leqq 2\pi$ are in one-to-one correspondence.

From what precedes it follows that *to each value of* r *and* α *satisfying the conditions*

$$(10) \qquad 0 < r < \rho, \quad \alpha = \text{any value}$$

there corresponds one and but one pair of values s, t *satisfying the conditions*

$$0 < s < S_1, \quad t = \text{any value}$$

and the equations

$$(11) \qquad r \cos \alpha = \phi(s, t), \qquad r \sin \alpha = \psi(s, t).$$

The implicit functions

$$s = s(r, \alpha), \quad t = t(r, \alpha)$$

so defined are of class C' *in the neighborhood of any point* (r, α) *of the region* (10), since by (8) the determinant $\Delta = \phi_s\psi_t - \phi_t\psi_s$ of equations (11) is always different from zero when s and t satisfy the inequalities (9). It is easily seen that any two values of α differing by a multiple of 2π define with r the same values of s and t.

PRINCETON,
February, 1907.

SOME PARTICULAR SOLUTIONS IN THE PROBLEM OF n BODIES.

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LET the masses of n finite bodies be represented by $m_1, m_2, \cdots, m_n$. Suppose that the bodies lie always in the same plane, and that their coordinates with respect to their common center of mass as origin and a system of rectangular axes which rotate with the uniform angular velocity N are, respectively, $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$. Supposing that the bodies attract each other according to the newtonian law, the differential equations of motion are

$$(1) \qquad \begin{aligned} \frac{d^2x_i}{dt^2} - 2N\frac{dy_i}{dt} - N^2x_i &= -\sum_{j=1}^{n} \frac{m_j(x_i - x_j)}{r^3_{i,j}}, \\ \frac{d^2y_i}{dt^2} + 2N\frac{dx_i}{dt} - N^2y_i &= -\sum_{j=1}^{n} \frac{m_j(y_i - y_j)}{r^3_{i,j}}, \end{aligned}$$

$$r_{i,j} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \qquad \begin{pmatrix} i = 1, 2, \cdots, n, \\ j \neq i \end{pmatrix}.$$

If we assume that each body is revolving in a circle about the common center of mass of the system with the uniform angular velocity N, its coordinates with respect to the rotating axes are constants, and the derivatives of the coordinates with respect to the time are zero. Equations (1) therefore reduce to the following system of algebraic equations:

$$(2) \qquad \begin{aligned} -N^2 x_i + \sum_{j=1}^{n} \frac{m_j(x_i - x_j)}{r_{i,j}^3} &= 0, \\ -N^2 y_i + \sum_{j=1}^{n} \frac{m_j(y_i - y_j)}{r_{i,j}^3} &= 0 \end{aligned} \qquad \begin{pmatrix} i = 1, 2, \cdots, n, \\ j \neq i \end{pmatrix}.$$

It is to be observed that the equations

$$(3) \qquad \begin{aligned} m_1 x_1 + m_2 x_2 + \cdots + m_n x_n &= 0, \\ m_1 y_1 + m_2 y_2 + \cdots + m_n y_n &= 0, \end{aligned}$$

which express the fact that the origin of coordinates is at the center of mass, follow from the system (2), and may be used instead of two of equations (2).

This system of $2n$ simultaneous algebraic equations involves the square of the angular velocity N, the $n - 1$ ratios of the masses, and the $2n - 1$ ratios of the distances x_i and y_i. Accordingly $n - 1$ of these quantities may be chosen arbitrarily and, if the resulting equations are independent, the remaining $2n$ quantities are determined by the relations (2). In order to admit physical interpretation, the quantity N^2 and the masses must be real and positive, while the coordinates must be real. With these restrictions it is not easy to discuss the general solutions of the system (2). Some interesting results may, however, be obtained by a study of special cases. If, in the problem of three bodies, the assumption is made that the triangle formed by the three bodies is isosceles, it may be shown that equations (2) can be satisfied only if the triangle is also equilateral, which is the well-known solution of Lagrange.

A particular case of the problem of four bodies will be considered in detail. The system of equations to be satisfied is

$$
(4)\qquad
\begin{aligned}
&m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 = 0, && (a)\\
&-N^2x_1 + \frac{m_2(x_1-x_2)}{r^3_{1,2}} + \frac{m_3(x_1-x_3)}{r^3_{1,3}} + \frac{m_4(x_1-x_4)}{r^3_{1,4}} = 0, && (b)\\
&-N^2x_2 + \frac{m_1(x_2-x_1)}{r^3_{2,1}} + \frac{m_3(x_2-x_3)}{r^3_{2,3}} + \frac{m_4(x_2-x_4)}{r^3_{2,4}} = 0, && (c)\\
&-N^2x_3 + \frac{m_1(x_3-x_1)}{r^3_{3,1}} + \frac{m_2(x_3-x_2)}{r^3_{3,2}} + \frac{m_4(x_3-x_4)}{r^3_{3,4}} = 0, && (d)\\
&m_1y_1 + m_2y_2 + m_3y_3 + m_4y_4 = 0, && (e)\\
&-N^2y_1 + \frac{m_2(y_1-y_2)}{r^3_{1,2}} + \frac{m_3(y_1-y_3)}{r^3_{1,3}} + \frac{m_4(y_1-y_4)}{r^3_{1,4}} = 0, && (f)\\
&-N^2y_2 + \frac{m_1(y_2-y_1)}{r^3_{2,1}} + \frac{m_3(y_2-y_3)}{r^3_{2,3}} + \frac{m_4(y_2-y_4)}{r^3_{2,4}} = 0, && (g)\\
&-N^2y_4 + \frac{m_1(y_4-y_1)}{r^3_{4,1}} + \frac{m_2(y_4-y_2)}{r^3_{4,2}} + \frac{m_3(y_4-y_3)}{r^3_{4,3}} = 0, && (h)
\end{aligned}
$$

The assumption will be made that the bodies lie at the vertices of a parallelogram, and that opposite masses are equal,

$$m_1 = m_3 = m, \quad m_2 = m_4 = M.$$

Let the x-axis pass through m_1. Then the coordinates may be represented as follows:

$$m_1(\alpha, 0), \quad m_2(\beta, \gamma), \quad m_3(-\alpha, 0), \quad m_4(-\beta, -\gamma).$$

Equations (a) and (e) are satisfied. Equation (f) becomes $r^3_{1,2} = r^3_{1,4}$, whence $\beta = 0$, and the parallelogram must be a rhombus. Putting $\beta = 0$, equation (c) is satisfied. Equations (b) and (d) become identical, yielding

$$(5)\qquad N^2 = \frac{2M}{r^3_{1,2}} + \frac{2m}{r^3_{1,3}},$$

and equations (g) and (h) become identical, yielding

$$(6)\qquad N^2 = \frac{2m}{r^3_{1,2}} + \frac{2M}{r^3_{2,4}}.$$

The two values of N^2 thus found must be the same. Therefore the following relation determines the ratio of the masses when the ratio of the distances is given:

$$\frac{M}{m} = \frac{r_{2,4}^3(r_{1,3}^3 - r_{1,2}^3)}{r_{1,3}^3(r_{2,4}^3 - r_{1,2}^3)} = \frac{\gamma^3\{8\alpha^3 - (\sqrt{\alpha^2+\gamma^2})^3\}}{\alpha^3\{8\gamma^3 - (\sqrt{\alpha^2+\gamma^2})^3\}}. \tag{7}$$

The choice of the ratio of the distances is limited by the condition that the resulting ratio of the masses must be positive. This condition is expressed by the inequality

$$\frac{1}{\sqrt{3}} < \frac{\gamma}{\alpha} < \sqrt{3}. \tag{8}$$

Hence, in the problem of four bodies revolving in circles about their common center of mass with the same uniform angular velocity, the rhombus is a possible configuration provided the following conditions are fulfilled: (1) *Opposite masses are equal,* (2) *the ratio of the diagonals shall be greater than* $1/\sqrt{3}$ *and less than* $\sqrt{3}$. The ratio of the masses is then *uniquely* determined by equation (7), and the angular velocity by equation (5) or equation (6).

Suppose a fifth body is introduced at the center of the rhombus. What will be the effect upon the limits within which one may choose the ratio of the diagonals? The method of treatment is analogous to that which precedes. The number of equations to be satisfied is ten, which, under the assumption that the configuration is a rhombus with opposite masses equal, reduce to the following two:

$$\begin{aligned} N^2 &= \frac{2M}{(\sqrt{\alpha^2+\gamma^2})^3} + \frac{2m}{8\alpha^3} + \frac{m_5}{\alpha^3}, \\ N^2 &= \frac{2m}{(\sqrt{\alpha^2+\gamma^2})^3} + \frac{2M}{8\gamma^3} + \frac{m_5}{\gamma^3}. \end{aligned} \tag{9}$$

Taking m_5 as the unit of mass, there results the following condition upon the masses m and M and the ratio γ/α of the diagonals of the rhombus:

$$M = \left\{\frac{\gamma^3\{8\alpha^3 - (\sqrt{\alpha^2+\gamma^2})^3\}}{\alpha^3\{8\gamma^3 - (\sqrt{\alpha^2+\gamma^2})^3\}}\right\} m + \frac{4(\alpha^3-\gamma^3)(\sqrt{\alpha^2+\gamma^2})^3}{\alpha^3\{8\gamma^3 - (\sqrt{\alpha^2+\gamma^2})^3\}}.$$

Regarding γ/α as a parameter, this equation represents a straight line in the mM-plane. Only those pairs of values

(m, M) are admissible which represent a point in the first quadrant. Hence γ/α must be selected so that some portion of the straight line lies in the first quadrant. This condition will certainly be satisfied if the slope is positive, that is, if the coefficient of m is positive. This condition is

$$\frac{1}{\sqrt{3}} < \frac{\gamma}{\alpha} < \sqrt{3},$$

which is the same as the one found in the case of four bodies. If, in the case of four bodies, the ratio $\gamma/\alpha = 1/\sqrt{3}$ (or $\sqrt{3}$) be selected the ratio (M/m) of the masses is infinite (or zero). If, in the case of five bodies, the ratio $\gamma/\alpha = 1/\sqrt{3}$ (or $\sqrt{3}$) be selected there results $m = -4$ (or $M = -4$), which are not admissible. If the slope of the line is negative, it may still lie partly in the first quadrant if the intercept on the M-axis is positive. It is easily verified, however, that values of the ratio γ/α which render the slope negative make the intercept negative. Hence the limits of the ratio of the diagonals are the same in the case of five bodies as in the case of four. In the latter case the ratio of the masses is *uniquely* determined by the choice of the ratio of the diagonals. In the case of five bodies there is a single infinitude of possibilities (except for $\gamma/\alpha = 1$, *i. e.*, when the rhombus becomes a square).

The rhombus, which is a special case of the parallelogram, is also a special case of the contraparallelogram.* It is interesting to notice, in the general case, that there exists a condition on the distances independent of the masses. The contraparallelogram is obtained by making $r_{1,2} = r_{1,4}$, $r_{3,2} = r_{3,4}$, $m_2 = m_4$. Under these assumptions the coordinates may be represented as follows: $m_1(\alpha, 0)$, $m_2(\beta, \gamma)$, $m_3(-\delta, 0)$, $m_4(\beta, -\gamma)$. The equations of condition (4) reduce to the following set:

$$m_1\alpha + 2m_2\beta - m_3\delta = 0,$$

$$N^2\alpha = \frac{2m_2(\alpha - \beta)}{r_{1,2}^3} + \frac{m_3(\alpha + \delta)}{r_{1,3}^3},$$

$$(10) \qquad N^2\beta = \frac{m_1(\beta - \alpha)}{r_{1,2}^3} + \frac{m_3(\beta + \delta)}{r_{2,3}^3},$$

$$N^2\gamma = \frac{m_1}{r_{1,2}^3} + \frac{2m_2}{r_{2,4}^3} + \frac{m_3}{r_{2,3}^3}.$$

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* Cf. Andoyer, "Sur l'équilibre relatif de n corps," *Bull. astron.*, vol. 23, pp. 50–59 (Feb., 1906).

Eliminating N^2 between the second and third of equations (10) there results the following condition upon the distances:

$$(11) \qquad \frac{\delta(\alpha-\beta)}{r_{1,2}^3}+\frac{\beta(\alpha+\delta)}{r_{1,3}^3}-\frac{\alpha(\beta+\delta)}{r_{2,3}^3}=0.$$

If equation (11) is satisfied, equations (10) furnish a system of two linear equations for the determination of the ratios of the masses. Hence in any case in which the solution is possible the determination of the masses is unique. If the conditions upon the distances can be satisfied, the fourth of equations (10) insures a real angular velocity.

Symmetric Configurations. In the case of three bodies at the vertices of an equilateral triangle there is no restriction upon the masses; they may be equal. In the case of four bodies at the vertices of a square the masses must be equal. In the case of n bodies it is known that, if all the bodies are equal, the regular polygon of n vertices is a possible configuration. It is known * also that if n equal masses are arranged at the vertices of a regular polygon, n other equal masses may be arranged at the vertices of another regular polygon, such that every mass of the second set lies upon a bisector of a central angle of the first polygon; that is, the $2n$ masses are arranged upon the circumferences of two concentric circles. The configuration is symmetric with respect to the line joining any one of the bodies to the center and the masses are distributed symmetrically with respect to this line. A configuration possessing these properties may be called a *symmetric configuration.*

For the consideration of symmetric configurations it is convenient to employ polar coordinates. The equations of condition, analogous to the system (2), are

$$(12) \qquad -N^2 r_i+\sum_{j=1}^{n}\frac{m_j\{r_i-r_j\cos(\theta_i-\theta_j)\}}{r_{i,j}^3}=0,$$

$$(13) \qquad \sum_{j=1}^{n}\frac{m_j r_j\sin(\theta_i-\theta_j)}{r_{i,j}^3}=0 \quad \begin{pmatrix} i=1,2,\cdots,n, \\ j\neq i \end{pmatrix}.$$

It is obvious that for any symmetric configuration the equations (13), involving $\sin(\theta_i - \theta_j)$, are satisfied. Some of the equations involving $\cos(\theta_i - \theta_j)$ become identical, and the number of independent equations to be satisfied is consequently reduced. Suppose the number of masses is $n = kl$, arranged on l concentric circles; that is, upon each circle are k equal masses. As i takes the values $1, 2, \cdots, k$, corresponding to the bodies on the circle of radius r_1, the resulting equations are identical; as i takes values corresponding to the k masses on the circle of radius r_2, the equations are identical; etc. Therefore the system (12) reduces to l independent equations involving N^2, the $l-1$ ratios of the distances, and the $l-1$ ratios of the masses. When N^2 is eliminated from this system of l equations there results a system of $l-1$ equations, linear and homogeneous in the masses, the coefficients being functions of the radii. The distances must be chosen so that the masses are positive. If the necessary conditions upon the distances can be satisfied, the ratios of the masses are *uniquely* determined by the solution of a linear system of equations.

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* Cf. R. Hoppe, "Erweiterung der bekannten Speciallösung des Dreikörperproblems." *Archiv der Math. und Phys.*, vol. 64, pp. 218–23.

As a first example suppose $n = 8$. Four of the masses ($m_1 = m_2 = m_3 = m_4 = m$) are arranged on a circle of radius r, and four ($m_5 = m_6 = m_7 = m_8 = M$) upon a circle of radius R. The configuration may be made symmetric in two ways,

$$1)\quad \theta_1 = \theta_5 = 0,\quad \theta_2 = \theta_6 = \frac{\pi}{2},\quad \theta_3 = \theta_7 = \pi,\quad \theta_4 = \theta_8 = \frac{3\pi}{2}.$$

$$2)\quad \begin{array}{llll} \theta_1 = 0, & \theta_2 = \dfrac{\pi}{2}, & \theta_3 = \pi, & \theta_4 = \dfrac{3\pi}{2}, \\ \theta_5 = \dfrac{\pi}{4}, & \theta_6 = \dfrac{3\pi}{4}, & \theta_7 = \dfrac{5\pi}{4}, & \theta_8 = \dfrac{7\pi}{4}. \end{array}$$

1) (Fig. 1). The equations of condition (12) reduce to the following set:

$$(14)\quad \begin{aligned} N^2 r &= \frac{2mr}{(\sqrt{2}r)^3} + \frac{2mr}{(2r)^3} + \frac{M(r-R)}{(R-r)^3} + \frac{2Mr}{(\sqrt{r^2+R^2})^3} + \frac{M(r+R)}{(r+R)^3}, \\ N^2 R &= \frac{m(R-r)}{(R-r)^3} + \frac{2mR}{(\sqrt{r^2+R^2})^3} + \frac{m(R+r)}{(R+r)^3} + \frac{2MR}{(\sqrt{2}R)^3} + \frac{2MR}{(2R)^3}. \end{aligned}$$

Selecting r for the unit of distance and m for the unit of mass, there results the following equation for the determination of M when R is given:

$$M\left\{\frac{1}{\sqrt{2}R^3}+\frac{1}{4R^3}+\frac{1}{(R-1)^2}-\frac{2}{(\sqrt{1+R^2})^3}-\frac{1}{(1+R)^2}\right\}\tag{15}$$
$$=\frac{1}{\sqrt{2}}+\frac{1}{4}-\frac{1}{R(R-1)^2}-\frac{2}{(\sqrt{1+R^2})^3}-\frac{1}{R(1+R)^2}.$$

If $R=2$, $M=0.2636$; if $R=5$, $M=9.5914$; if $R=10$, $M=308.814$.

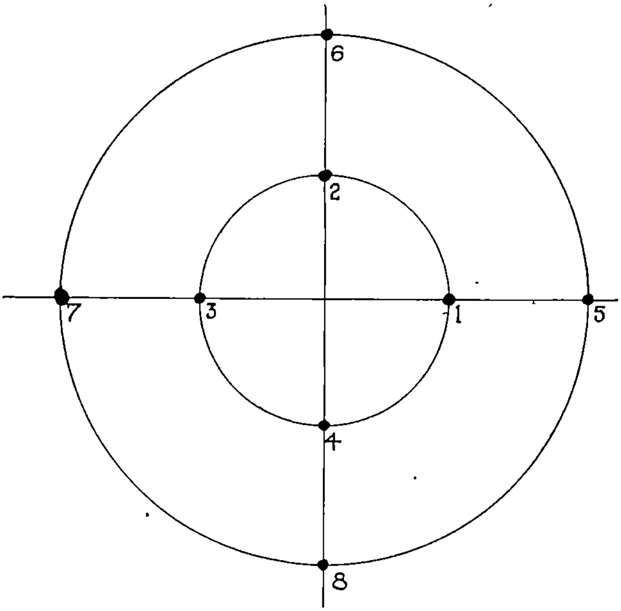

FIG. 1.

2) (Fig. 2). The equations of condition (12) reduce to the following set:

$$N^2r=\frac{2mr}{(\sqrt{2}r)^3}+\frac{2mr}{(2r)^3}+\frac{2M\left(r-\frac{R}{\sqrt{2}}\right)}{(\sqrt{r^2+R^2-\sqrt{2}rR})^3}+\frac{2M\left(r+\frac{R}{\sqrt{2}}\right)}{(\sqrt{r^2+R^2+\sqrt{2}rR})^3},\tag{16}$$
$$N^2R=\frac{2m\left(R-\frac{r}{\sqrt{2}}\right)}{(\sqrt{r^2+R^2-\sqrt{2}rR})^3}+\frac{2m\left(R+\frac{r}{\sqrt{2}}\right)}{(\sqrt{r^2+R^2+\sqrt{2}rR})^3}+\frac{2MR}{(\sqrt{2}R)^3}+\frac{2MR}{(2R)^3}.$$

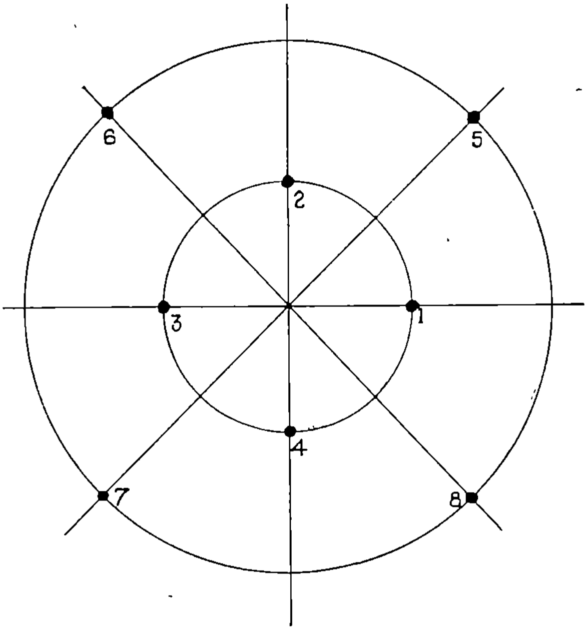

FIG. 2.

Selecting r for the unit of distance and m for the unit of mass, there results the following equation for the determination of M when R is given:

$$(17)\quad \begin{aligned} &M\left\{\frac{1}{\sqrt{2}R^3}+\frac{1}{4R^3}+\frac{R\sqrt{2}-2}{(\sqrt{1+R^2-R\sqrt{2}})^3}-\frac{R\sqrt{2}+2}{(\sqrt{1+R^2+R\sqrt{2}})^3}\right\} \\ &=\frac{1}{\sqrt{2}}+\frac{1}{4}-\frac{2R-\sqrt{2}}{R(\sqrt{1+R^2-R\sqrt{2}})^3}-\frac{2R+\sqrt{2}}{R(\sqrt{1+R^2+R\sqrt{2}})^3}. \end{aligned}$$

If $R=2$, $M=2.622$; if $R=5$, $M=10.197$; if $R=10$, $M=330.37$.

As a second example suppose $n=8$ and that the masses are arranged on three circles as indicated in Fig. 3. Suppose

$$m_1=m_2=m,\quad m_3=m_4=M,\quad m_5=m_6=m_7=m_8=M',$$

$$r_1=r_2=r,\quad r_3=r_4=R,\quad r_5=r_6=r_7=r_8=R'.$$

The equations of condition (12) reduce to the following set:

$$
(18)\quad
\begin{aligned}
N^2r &= \frac{2mr}{(2r)^3} + \frac{2Mr}{(\sqrt{r^2+R^2})^3} + \frac{2M'r}{(\sqrt{r^2+R'^2})^3} \\
&\qquad + \frac{M'(r-R')}{(R'-r)^3} + \frac{M'(r+R')}{(R'+r)^3}, \\
N^2R &= \frac{2mR}{(\sqrt{r^2+R^2})^3} + \frac{M2R}{(2R)^3} + \frac{2M'R}{(\sqrt{R^2+R'^2})^3} \\
&\qquad + \frac{M'(R-R')}{(R'-R)^3} + \frac{M'(R+R')}{(R+R')^3}, \\
N^2R' &= \frac{m(R'-r)}{(R'-r)^3} + \frac{m(R'+r)}{(R'+r)^3} + \frac{M'(2R')}{(2R')^3}, \\
&\qquad + \frac{2MR'}{(\sqrt{R^2+R'^2})^3} + \frac{2M'R'}{(\sqrt{2}R')^3}.
\end{aligned}
$$

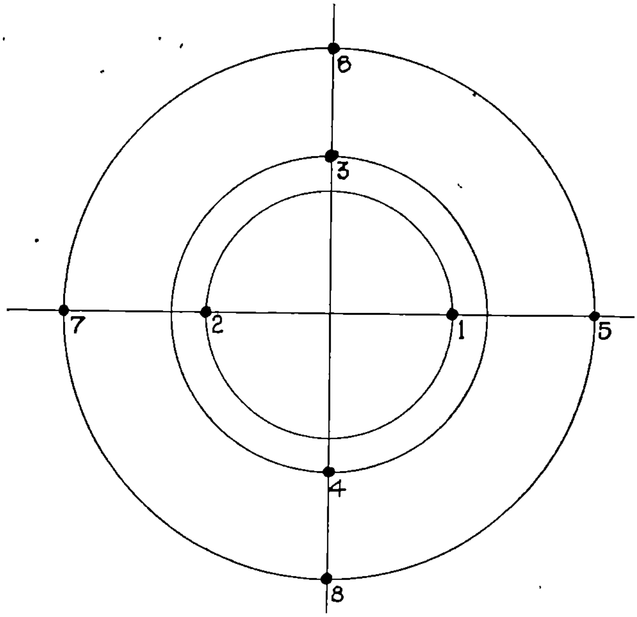

FIG. 3.

Putting $r = 1$, $m = 1$, the following system of equations determines M and M' when R and R' are given:

$$(19) \qquad aM + bM' = c, \quad a_1M + b_1M' = c_1,$$

where

$$a = \frac{2}{(\sqrt{1+R^2})^3} - \frac{1}{4R^3},$$

$$b = \frac{2}{(\sqrt{1+R'^2})^3} - \frac{1}{(R'-1)^2} + \frac{1}{(R'+1)^2}$$

$$- \frac{2}{(\sqrt{R^2+R'^2})^3} + \frac{1}{R(R'-R)^2} - \frac{1}{R(R'+R)^2},$$

$$(20) \quad c = \frac{2}{(\sqrt{1+R^2})^3} - \frac{1}{4},$$

$$a_1 = \frac{2}{(\sqrt{1+R^2})^3} - \frac{2}{(\sqrt{R^2+R'^2})^3},$$

$$b_1 = \frac{2}{(\sqrt{1+R'^2})^3} - \frac{1}{(R'-1)^2} + \frac{1}{(R'+1)^2} - \frac{1}{4R'^3} - \frac{1}{\sqrt{2}R'^3},$$

$$c_1 = \frac{1}{R'(R'-1)^2} + \frac{1}{R'(R'+1)^2} - \frac{1}{4}.$$

If $R = 1.2$, $R' = 5$; $M = 0.6331$, $M' = 20.332$.

If $R = 1.3$, $R' = 5$; $M = 0.4768$, $M' = 16.081$.

If $R = 1.4$, $R' = 5$; $M = 0.3140$, $M' = 12.848$.

If $R = 1.5$, $R' = 5$, the resulting value of M is negative. In this case, also, the ratio of the diagonals of the rhombus formed by the inner bodies is limited, the upper limit lying between 1.4 and 1.5.

The preceding discussion refers to configurations which are fixed in size, so that each body revolves about the center of mass of the system in a circle. The results can be extended at once to the case where the configuration preserves always the same shape, but varies in size so that each body describes an ellipse with the center of mass at one focus. As an example, consider the case of four bodies. The differential equations of motion in polar coördinates are

$$(21)\qquad \begin{gathered} \frac{d^2r_i}{dt^2} - r_i\left(\frac{d\theta_i}{dt}\right)^2 = -\sum_{j=1}^{4}\frac{m_j\{r_i - r_j\cos(\theta_i - \theta_j)\}}{r^3_{i,j}}, \\ r_i\frac{d^2\theta_i}{dt^2} + 2\frac{dr_i}{dt}\frac{d\theta_i}{dt} = -\sum_{j=1}^{4}\frac{m_j r_j \sin(\theta_i - \theta_j)}{r^3_{i,j}} \end{gathered}$$

$$(i = 1, 2, 3, 4;\ j \neq i).$$

Suppose the bodies lie at the vertices of a rhombus, which may vary in size, but in which the ratio ρ of the diagonals is constant, and selected within the limits $1/\sqrt{3} < \rho < \sqrt{3}$. Suppose also that opposite masses are equal, $m_1 = m_3 = m$; $m_2 = m_4 = M$. Suppose that the ratio of the masses is determined by the relation (7) $[\gamma/\alpha = \rho]$. We may make the followlowing conditions upon the coordinates:

$$\theta_1 = \theta, \quad \theta_2 = \theta + \frac{\pi}{2}, \quad \theta_3 = \theta + \pi, \quad \theta_4 = \theta + \frac{3\pi}{2},$$

$$r_1 = r_3 = r, \quad r_2 = r_4 = \rho r.$$

The set of equations (21) now reduces to the set (22) for the determination of r and θ,

$$(22)\qquad \begin{gathered} \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{r^2}\left\{\frac{2M}{(1+\rho^2)^{\frac{3}{2}}} + \frac{m}{4}\right\} = 0, \\ r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = 0. \end{gathered}$$

These are the equations for elliptic motion. Let N = mean angular motion, and A = major semi-axis. We have the relation

$$N^2A^3 = \frac{2M}{(1+\rho^2)^{\frac{3}{2}}} + \frac{m}{4}.$$

This is the analogue of the relation $N^2A^3 = m_1 + m_2$, which holds in the problem of two bodies.

YALE UNIVERSITY,
December, 1906.

ON THE MATRICES OF PERIOD A POWER OF p IN JORDAN'S LINEAR CONGRUENCE GROUPS, MODULO p^a.

BY DR. ARTHUR RANUM.

(Read before the American Mathematical Society, October 27, 1906.)

Introduction.

IN contrasting Jordan's * linear congruence groups, modulo p^a, with the more general "groups of classes of congruent matrices, modulo $(p^{a_{ij}})$", which I have recently discovered,† it appears that in general the latter groups cannot be abstractly identical with any of the former groups of the same degree or subgroups thereof, and are therefore new groups of that degree, abstractly as well as concretely. The proof of this assertion given in the paper just referred to depends for its validity on certain properties of Jordan groups to be derived in this note. These properties have to do with the matrices of period a power of p. Moreover a formula is found for the νth power of any matrix.

The νth Power of a Matrix.

1. Consider any n-ary matrix $M = (m_{ij})$, where m_{ij} $(i, j = 1, \cdots, n)$ is the element in the ith row and jth column. Denote its νth power by $M^\nu = (m_{ij}^{(\nu)})$, where ν is a positive integer. Then $m_{ij}^{(1)} = m_{ij}$ $(i, j = 1, \cdots, n)$.

For many purposes, including the application to Jordan groups that is to follow, it is convenient to put $m_{ii} = 1 + l_{ii}$, $m_{ij} = l_{ij}$ $(i, j = 1, \cdots, n;\ i \neq j)$, *i. e.*, to write M in the form

$$\begin{pmatrix} 1 + l_{11} & l_{12} & \cdots \\ l_{21} & 1 + l_{22} & \cdots \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix},$$

and to express its νth power in terms of the partial elements l_{ij} $(i, j = 1, \cdots, n)$. If the symbolic addition and subtraction, as well as multiplication, of matrices be defined in the usual way, and if the special matrix (δ_{ij}), where $\delta_{ij} = \left\{\begin{smallmatrix}1, \text{ if } i=j\\ 0, \text{ if } i\neq j\end{smallmatrix}\right.$, be denoted by I, then we have $(m_{ij}) = (\delta_{ij}) + (l_{ij})$, *i. e.*, $M = I + L$. Therefore

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* Jordan, Traité des substitutions (1870), pp. 91–110, and in particular pp. 103–105.

† *Transactions Amer. Math. Society*, vol. 8, No. 1 (Jan., 1907), pp. 71–91.

$$M^{\nu} = (I + L)^{\nu} = I + \sum_{\mu=1}^{\nu} \binom{\nu}{\mu} L^{\mu}, \tag{1}$$

which is equivalent to the n^2 equations

$$m_{ij}^{(\nu)} = \delta_{ij} + \sum_{\mu=1}^{\nu} \binom{\nu}{\mu} l_{ij}^{(\mu)} \qquad (i, j = 1, \cdots, n), \tag{2}$$

by means of which the elements of M^{ν} are expressed in terms of the corresponding elements of the powers of L from the first up to the νth. Moreover, since every element of L^{μ} is a homogeneous function, of degree μ, of the elements of L, every element $m_{ij}^{(\nu)}$ of M^{ν} is a non-homogeneous function, of degree ν, of the partial elements l_{ij} $(i, j = 1, \cdots, n)$ of M, and its terms of degree μ are together equal to $\binom{\nu}{\mu} l_{ij}^{(\mu)}$.

2. To find these terms of degree μ, *i. e.*, to express L^{μ} $(\mu = 2, \cdots, \nu)$ in terms of L, there are at least two distinct methods of procedure.

(*a*) Since $L^{\mu} = LL^{\mu-1}$, we have the recursion formulas

$$l_{ij}^{(\mu)} = \sum_{k=1}^{n} l_{ik} l_{kj}^{(\mu-1)} \qquad \left(\begin{matrix} i, j = 1, \cdots, n, \\ \mu = 2, \cdots, \nu \end{matrix}\right), \tag{3}$$

by means of which every element of L^{μ} is expressed in terms of the elements of a single row of L and a single column of $L^{\mu-1}$.

(*b*) Putting $L^0 = I$ and denoting by d_r $(r = 1, \cdots, n)$ the sum of the principal $(n - r)$th minors of the determinant of L, so that d_n is the determinant itself, we can write the identical equation * that L satisfies in the form

$$L^n = \sum_{r=1}^{n} (-1)^{r-1} d_r L^{n-r}.$$

If $\mu \geqq n$, we multiply this equation by $L^{\mu-n}$ $(\mu = n, \cdots, \nu)$ and derive the equations

$$L^{\mu} = \sum_{r=1}^{n} (-1)^{r-1} d_r L^{\mu-r} \quad (\mu = n, n+1, \cdots, \nu), \tag{4}$$

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* Cayley, "Collected mathematical papers," vol. 2, pp. 482–492 Frobenius, *Crelle's Journal*, vol. 84, pp. 1–16. Taber, *Amer. Jour. of Mathematics*, vol. 12 (1890), pp. 357, 358.

which are equivalent to the recursion formulas *

$$(5) \qquad l_{ij}^{(\mu)} = \sum_{r=1}^{n}(-1)^{r-1} d_r l_{ij}^{(\mu-r)} \qquad \left(\begin{matrix} i, j = 1, \cdots, n, \\ \mu = n, n+1, \cdots, \nu \end{matrix}\right),$$

in which d_r $(r = 1, \cdots, n)$ is independent of i, j and μ. By means of (5) every element of L_μ $(\mu \geqq n)$ is expressed in terms of the corresponding elements of the n next lower powers of L and the principal minors of the determinant of L.

3. By repeated applications of (3) or (5) all the powers of L from L^2 to L^ν can be computed, except that if $n > 2$, (3) must be used to find the first $n - 1$ powers, after which either (3) or (5) can be used to find the rest. Finally, the substitution of the values so obtained in (2) gives the νth power of M in the required form.

Jordan Groups: Certain Invariant Subgroups.

4. Let G be an n-ary Jordan group, or linear congruence group, modulo p^a, where p is a prime and a is a positive integer, *i. e.*, G comprises the totality of the n-ary matrices, whose elements are residues of p^a and whose determinants are prime to p, these matrices combining with each other by multiplication. In G consider the invariant subgroup H_b, whose matrices are congruent to the matrix I, modulo p^b $(b \leqq a)$, and are therefore of the form $I + p^b L$. Clearly $H_0 = G$ itself, and $H_a = I$. If $b > 0$, H_b is of order $p^{n^2(a-b)}$ and its matrices are all of period a power of p. These groups $H_0, H_1, \cdots, H_a$ form a series, each of which contains the next in order and therefore all that follow it.

5. If, in order to indicate the fundamental modulus p^a, we introduce the additional subscript a and write G_a for G and $H_{b,a}$ for H_b, then it is clear that $G_a / H_{b,a} = G_b$ $(b = 0, \cdots, a)$, and, more generally, that $H_{b,a} / H_{c,a} = H_{b,c}$ $(b, c = 0, \cdots, a; b \leqq c)$.

6. THEOREM 1.† *The commutator of a matrix of H_b and a matrix of H_c is a matrix of H_{b+c}, if $b + c \leqq a$; while if $b + c \geqq a$, every matrix of H_b is commutative with every matrix of H_c.*

Proof. Let $L_b = I + p^b L$ and $M_c = I + p^c M$ be any two matrices of H_b and H_c respectively. Then

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* For the cases $n = 2$ and $n = 3$ these formulas were derived by direct calculation, without the use of the identical equation, by Weltzien, "Ueber Produkte und Potenzen von Determinanten," Berlin, 1897.

† The first part of this theorem was proved by Jordan for the case $c = 1$, Traité des substitutions, p. 104.

$$L_bM_c \equiv I + p^bL + p^cM + p^{b+c}LM, \quad \text{mod } p^a,$$

and

$$M_cL_b \equiv I + p^bL + p^cM + p^{b+c}ML, \quad \text{mod } p^a.$$

Therefore, if $b + c \leqq a$,

$$L_bM_c \equiv M_cL_b, \quad \text{mod } p^{b+c},$$

i. e.,

$$L_b^{-1}M_c^{-1}L_bM_c \equiv I, \quad \text{mod } p^{b+c},$$

which proves the theorem.

Corollaries. Putting $c = a - b$, we see that if $b \leqq a/2$, every matrix of H_{a-b} is invariant in H_b. Again, putting $c=b$, we see that if $b \leqq a/2$, all the commutators of H_b are contained in H_{2b}; while if $b \geqq a/2$, H_b is abelian. Therefore if a is even $(= 2a')$, $H_{a'}$ is abelian; while if a is odd $(= 2a' - 1)$, $H_{a'}$ is abelian.

7. Let the totality of the matrices of H_b that are not contained in its subgroup H_{b+1} be denoted by $H_b - H_{b+1}$. Supposing $a > 1$, consider the subgroups $H_1, \cdots, H_a$, leaving H_0 out of account.

THEOREM 2. *Every matrix of* $H_b - H_{b+1} (b = 1, \cdots, a - 1)$* *is of period* p^{a-b}, *except when* $b = 1$ *and* $p = 2$.

Proof. Let $L_b = I + p^bL$ be any matrix of H_b. Then by (1),

$$L_b^{p^{a-b}} \equiv I, \quad \text{mod } p^a,$$

and the period of L_b is a divisor of p^{a-b}. Moreover,

$$L_b^{p^{a-b-1}} \equiv I + p^{a-1}L + \binom{p^{a-b-1}}{2} p^{2b}L^2 + \cdots, \quad \text{mod } p^a,$$

where the remaining terms are all divisible by p^a, and, except when $b = 1$ and $p = 2$, the third term also; so that the congruence reduces to the form

$$L_b^{p^{a-b-1}} \equiv I + p^{a-1}L, \quad \text{mod } p^a.$$

Now if the period of L_b is less than p^{a-b}, L is divisible by p and L_b is contained in H_{b+1}. This proves the theorem.

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* A simple example, for $n = 2$, is $\left(\begin{smallmatrix} 1 & 0 \\ p^b & 1 \end{smallmatrix}\right)$.

Moreover, it is easy to see that if $p = 2$, $H_1 - H_2$ contains matrices not only of period 2^{a-1}, but also of periods $2^{a-2}, \cdots, 2$.*

Jordan Groups: Sylow Subgroups of Order a Power of p.

8. In the n-ary Jordan group G, modulo p^a, consider the totality of matrices of the form

$$M = \begin{pmatrix} 1 + p\lambda_{11} & p\lambda_{12} & \cdots & p\lambda_{1n} \\ \lambda_{21} & 1 + p\lambda_{22} & \cdots & p\lambda_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \lambda_{n1} & \lambda_{n2} & \cdots & 1 + p\lambda_{nn} \end{pmatrix} = I + L,$$

in which

$$(6) \qquad l_{ij} \equiv 0, \quad \mathrm{mod}\, p \quad (i, j = 1, \cdots, n;\ i - j \leqq 0).$$

It is easy to verify that they form a group of order $p^{n^2(a-1)+\frac{1}{2}n(n-1)}$ and therefore a Sylow subgroup of G.† Call it H and its conjugates H', H'', etc. The greatest common subgroup of H, H', etc., is evidently H_1, the periods of whose matrices have been determined in § 7.

9. It remains to find the periods of the matrices of $H - H_1$. In order to raise M to the νth power by means of (1) and (2), it will be necessary to find the p-factors (powers of p that are factors) of the elements of the different powers of L.

From (6) we derive by induction the formula

$$(7) \qquad l_{ij}^{(\mu)} \equiv 0, \quad \mathrm{mod}\, p \quad (i, j = 1, \cdots, n;\ i - j \leqq \mu - 1).$$

For, by (3), we have

$$l_{ij}^{(\mu+1)} = \sum_{k=1}^{n} l_{ik} l_{kj}^{(\mu)},$$

every term of which, in view of $(6)_{ik}$ and $(7)_{kj}$, is divisible by p, if $i - j \leqq \mu$. Therefore (7) holds for all values of μ.

Moreover,

$$l_{1n}^{(2)} = \sum_{k=1}^{n} l_{1k} l_{kn}$$

---

* E. g., the binary matrix $\begin{pmatrix} -1 & 0 \\ 2^b & -1 \end{pmatrix}$, of period $2^{a-b} (b = 1, \cdots, a - 1)$.

† This fact was pointed out to me by Professor Dickson.

is divisible by p^2. From this fact, by induction, we derive the formula

(8) $l_{ij}^{(\mu)} \equiv 0, \bmod p^2 \quad (\mu \geqq 2; i, j = 1, \cdots, n; i-j \leqq \mu-1-n).$

For we have

$$l_{ij}^{(\mu+1)} = \sum_{k=1}^{i-1} l_{ik} l_{kj}^{(\mu)} + \sum_{k=i}^{n} l_{ik} l_{kj}^{(\mu)},$$

which is divisible by p^2, if $i - j \leqq \mu - n$.

The more general formula

(9) $$l_{ij}^{(\mu)} \equiv 0, \quad \bmod p^{b+1}$$
$$[\mu \geqq (b-1)n + 2;\ i, j = 1, \cdots, n,\ i - j \leqq \mu - 1 - bn),$$

of which (7) and (8) are special cases, can be proved in a similar manner.

Since $i - j$ is always less than n, (9) gives rise to the formula

(10) $$l_{ij}^{(\mu)} \equiv 0, \quad \bmod p^{b}$$
$$(b = 1, 2, \cdots;\ \mu = bn, bn+1, \cdots;\ i, j = 1, \cdots, n);$$

i. e., every element of L^{bn}, L^{bn+1}, $\cdots$ is divisible by p^b.

10. If M be chosen such that

(11) $$l_{ij} \equiv 0, \quad \bmod p \quad (i, j = 1, \cdots, n;\ i - j \leqq k - 1),$$

then by a similar process the formulas

(12) $$l_{ij}^{(\mu)} \equiv 0, \quad \bmod p \quad (i, j = 1, \cdots, n;\ i - j \leqq k\mu - 1),$$

and

(13) $$l_{ij}^{(\mu)} \equiv 0, \quad \bmod p^2$$
$$[\mu \geqq 2;\ i, j = 1, \cdots, n;\ i - j \leqq k(\mu - 1) - n],$$

can be derived.

11. We are now prepared to prove

THEOREM 3. *If $p > n - 1$, p^a is the highest power of p that is the period of any matrix of the n-ary Jordan group, modulo p^a.*

Proof. Let $M = I + L$ be any matrix of H. Then

$$M^{p^a} \equiv I + \sum_{\mu=1}^{p^a} \binom{p^a}{\mu} L^{\mu}, \quad \bmod p^a.$$

Suppose that the G. C. D. of μ and p^a is p^b. Then it is clear that $\binom{p^a}{\mu}$ is divisible by p^{a-b}. Moreover, since $p \geqq n$, therefore $p^b \geqq bn$, and $\mu \geqq bn$. Consequently, by (10), every element of L^μ is divisible by p^b. Therefore, finally,

$$M^{p^a} \equiv I, \quad \bmod p^a;$$

i. e., the period of every matrix of H is a divisor of p^a. Moreover, matrices whose period is exactly p^a exist in $H - H_1$,[*] and the theorem is proved.

If $n = 2$, the condition $p > n - 1$ is always satisfied and the theorem is true for all values of p.

If $a = 1$, and $p > n - 1$, the theorem shows that every matrix of $H - I$ is of period p.

12. By a similar argument we derive the more general

THEOREM 4. *If* $p^b < n \leqq p^{b+1}$, *then* p^{a+b} *is the highest power of* p *that is the period of any matrix of the* n*-ary Jordan group, modulo* p^a.

Moreover, matrices of period $p^a, \cdots, p^{a+b}$ evidently exist in $H - H_1$.[†]

13. The question now arises whether $H - H_1$ also contains matrices of lower period than p^a.

THEOREM 5. *If* $p > n + 1$, *the period of every matrix of* $H - H_1$ *is exactly* p^a.

Proof. The proof consists in showing that if the period of a matrix M of H is less than p^a, then M must belong to H_1. Assume that $M^{p^{a-1}} \equiv I$, mod p^a. Since

$$M^{p^{a-1}} = I + \sum_{\mu=1}^{p^{a-1}} \binom{p^{a-1}}{\mu} L^\mu,$$

it follows that

$$\sum_{\mu=1}^{p^{a-1}} \binom{p^{a-1}}{\mu} L^\mu \equiv 0, \quad \bmod p^a,$$

which gives rise to the n^2 congruences

$$\sum_{\mu=1}^{p^{a-1}} \binom{p^{a-1}}{\mu} l_{ij}^{(\mu)} \equiv 0, \bmod p^a \quad (i, j = 1, \cdots, n). \tag{14}$$

Remembering that the elements l_{ij} satisfy (6), we wish to show, as a consequence of (14), that they are divisible by p for all values of i and j.

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[*] *E. g.*, if $n = 2$, the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$.

[†] *E. g.*, if $n = 3$, $p = 2$, $a = 1$, and therefore $b = 1$, the matrix $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{smallmatrix}\right)$, belonging to $H - I$, is of period 4.

In the first place, of the n^2 congruences (14), those for which $i-j \leqq 0$ are identically satisfied. For, by (7), $l_{ij}^{(\mu)}$ is in this case always divisible by p; moreover, if the G. C. D. of μ and p^{a-1} is p^b, $\binom{p^{a-1}}{\mu}$ is divisible by p^{a-b-1}, and since $p > 1+n$, $p^b > 1+bn$, so that, in view of (9), $l_{ij}^{(\mu)}$ $(i-j \leqq 0)$ is divisible by p^{b+1}.

Next, in those congruences (14) for which $i-j=1$, exactly the same reasoning applies to every term except the first. Therefore, those congruences reduce to the form $p^{a-1}l_{ij} \equiv 0$, mod p^a $(i-j=1)$, or $l_{ij} \equiv 0$, mod p $(i-j=1)$, which, combined with (6), becomes

$$l_{ij} \equiv 0, \bmod p \qquad (i-j \leqq 1). \tag{15}$$

If $n=2$, this proves the theorem. If $n>2$, this additional limitation on L makes it possible to reduce those congruences (14), for which $i-j=2, 3$, to a simple form. For, putting $k=2$ in (12), we see that $l_{ij}^{(\mu)} \equiv 0$, mod p $(\mu > 1;\ i-j=2, 3)$. Moreover, putting $k=2$ and $\mu=p$ in (13), and noting that since $p>n+1$, $2(p-1)-n>n \geqq 3$, we see that, if μ is divisible by p, $l_{ij}^{(\mu)}$ $(i-j=2, 3)$ is divisible by p^2. A fortiori, if μ is divisible by p^b, $l_{ij}^{(\mu)}$ $(i-j=2, 3)$ is divisible by p^{b+1}. Therefore those congruences (14), for which $i-j=2, 3$, reduce to the form $p^{a-1}l_{ij} \equiv 0$, mod p^a $(i-j=2, 3)$ or $l_{ij} \equiv 0$, mod p $(i-j=2, 3)$.

If $n=3$ or 4, this proves the theorem. If $n>4$, another application of (12) and (13), putting $k=4$, shows that $l_{ij} \equiv 0$, mod p $(i-j=4, 5, 6, 7)$; and by continuing this process we finally see that l_{ij} is divisible by p for all values of i and j, and therefore that the theorem is true for all values of n.

14. If $p \leqq n+1$, it is easy to see that theorem 5 is not true, and that $H-H_1$ contains matrices of period $p, p^2, \cdots, p^{a-1}$, as well as p^a. Indeed $G-H_1$ contains matrices of period $2, 3, \cdots, n, n+1$, independently of any modulus. *E. g.*, if $n=4$, $G-H_1$ contains the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

which is equivalent to the linear substitution $x_1' = x_2$, $x_2' = x_3$,

$x'_3 = x_4$, $x'_4 = x_1$, and is therefore of period 4, independently of any modulus. A fortiori, $G - H_1$ contains similar matrices of period 2 and 3. Moreover, the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{bmatrix},$$

also of $G - H_1$, is of period 5, independently of any modulus. For it is equivalent to the linear substitution $x'_1 = x_2$, $x'_2 = x_3$, $x'_3 = x_4$, $x'_4 = -x_1 - x_2 - x_3 - x_4$, and if x_0 be defined by the equation $x_0 + x_1 + x_2 + x_3 + x_4 = 0$, then the linear substitution is equivalent to the cyclic permutation $(x_0 x_1 x_2 x_3 x_4)$, which is of period 5.

Jordan Groups: Final Results.

15. By combining the results obtained in §§ 7, 11–14, we see that the periods of the matrices of $H - H_1$ are (*a*) $p, \cdots, p^a, \cdots, p^{a+b}$, if $p \leqq n-1$, (*b*) $p, \cdots, p^a$, if $p = n$ or $n+1$, (*c*) p^a, if $p > n+1$. Moreover, if we divide the matrices of period a power of p into the classes $\Sigma(H - H_1)$, $H_1 - H_2$, $H_2 - H_3$, $\cdots$, $H_{a-1} - \mathrm{I}$, where $\Sigma(H - H_1) = (H - H_1) + (H' - H_1) + \cdots$, we have the means of determining, for any given values of n and p, the exact range of the periods of the matrices in each class.

The following table gives these periods for a few small values of n and p.

		$\Sigma(H-H_1)$	H_1-H_2	H_2-H_3	$\ldots$	$H_{a-1}-\mathrm{I}$
$n=2$	$p=2$	$2^a, \cdot\ \cdot, 2$	$2^{a-1}, \cdots, 2$	2^{a-2}	$\ldots$	2
	$p=3$	$3^a, \cdot\cdot, 3$	3^{a-1}	3^{a-2}	$\ldots$	3
	$p=5$	5^a	5^{a-1}	5^{a-2}	$\ldots$	5
$n=5$	$p=2$	$2^{a+2}, 2^{a+1}, 2^a, \cdots, 2$	$2^{a-1}, \cdots, 2$	2^{a-2}	$\ldots$	2
	$p=3$	$3^{a+1}, 3^a, \cdot\cdot, 3$	3^{a-1}	3^{a-2}	$\ldots$	3
	$p=5$	$5^a, \cdot\ \cdot, 5$	5^{a-1}	5^{a-2}	$\ldots$	5
	$p=7$	7^a	7^{a-1}	7^{a-2}	$\ldots$	7

The groups of simplest structure are those in which $p > n+1$. In every such group we have an exhaustive classification of the matrices of period a power of p, *viz.*, those of period p^b ($b = 1, \cdots, a-1$) belong to $H_{a-b} - H_{a-b+1}$, and those of period p^a to $\Sigma(H - H_1)$.

16. As applied to these groups, Theorem 1 can now be stated in the following more significant form:

THEOREM 6. *In an n-ary Jordan group, modulo* p^a, *for which* $p > n + 1$, *the commutator of a matrix of period* p^b *and a matrix of period* p^c *is a matrix whose period is* p^{b+c-a} *or a lower power of* p, *if* $b + c \geqq a$, *while if* $b + c \leqq a$, *every matrix of period* p^b *is commutative with every matrix of period* p^c.

Under the same restriction that $p > n + 1$ we have the

Corollaries: In the subgroup generated by the matrices of period $p^b (b \geqq a/2)$, every matrix of period p^{a-b} is invariant. The commutator of any two matrices of period $p^b (b \geqq a/2)$ is a matrix whose period is p^{2b-a} or a lower power of p; while if $b \leqq a/2$, any two matrices of period p^b are commutative. If $b \leqq a/2$, the matrices whose periods are divisors of p^b form an abelian subgroup of G.

CORNELL UNIVERSITY,
December, 1906.

ON THE CONSTRUCTION OF AN INTEGRAL OF LAGRANGE'S EQUATIONS IN THE CALCULUS OF VARIATIONS.

BY DR. D. C. GILLESPIE.

(Read before the American Mathematical Society, December 29, 1906.)

1. DARBOUX [*] has proven the theorem:

When (1) $\int_1^2 F(y', y, x)dx$ and (2) $\int_1^2 F'(y', y, x)dx$

$$(y' = dy/dx)$$

lead to the same Lagrange equation; *i. e.*, when the first variation of integrals (1) and (2) equated to zero give the same differential equation (3) $y'' = \omega(y', y, x)$, where $y'' = d^2y/dx^2$, then $F_{y'y'}/F'_{y'y'} =$ const. (literal subscripts denote partial derivatives) is an intermediary of equation (3).

2. This theorem can be proven with very little algebraic work by making use of Hilbert's [†] independence theorem (Unabhängigkeitssatz).

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[*] Darboux, Théorie des surfaces, vol. 3, pp. 59–65.

[†] Hilbert's Vorlesungen über Variationsrechnung, Winter-Semester, 1904–1905.

This latter method lends itself readily to the investigation of the problem for a system of differential equations.

Let (5) $\int_1^2 F(z', z, y', y, x)dx$ and (6) $\int_1^2 F'(z', z, y', y, x)\,dx$,

be two integrals, the vanishing of the first variation of which leads to the same system of differential equations

$$(4) \qquad y'' = \omega_1(z', z, y', y, x), \quad z'' = \omega_2(z', z, y', y, x).$$

The purpose here is to show that

$$\begin{vmatrix} F_{y'y'}, & F_{y'z'} \\ F_{y'z'}, & F_{z'z'} \end{vmatrix} \div \begin{vmatrix} F'_{y'y'} & F'_{y'z'} \\ F'_{y'z'} & F'_{z'z'} \end{vmatrix} = \text{const.}$$

is an integral of the system (4) (in case of course the quotient itself is not identically constant).

The method of proof for a system of two differential equations gives at once the result for a system of m equations.

Out of the integrals (5) and (6) are built the integrals

$$(5') \quad \int_1^2 \left\{ F(q, z, p, y, x) + (y' - p)F_p + (z' - q)F_q \right\} dx,$$

$$(6') \quad \int_1^2 \left\{ F'(q, z, p, y, x) + (y' - p)F'_p + (z' - q)F'_q \right\} dx.$$

The method requires that there exist a $p(x, y, z, \alpha, \beta)$, $q(x, y, z, \alpha, \beta)$ (where α and β are arbitrary constants) which make integrals (5') and (6') independent of the path of integration. Although not every pair of functions $p(x, y, z,)$, $q(x, y, z)$ which render (5') independent of the path will necessarily render (6') independent of the path,* yet it follows from the results obtained by Mayer † that there exists a two-parameter system of functions $p(x, y, z, q, \alpha, \beta)$, $q(x, y, z, \alpha, \beta)$ which makes both integrals independent of the path of integration. We may assume therefore two such functions, p and q; it follows then that the integrand of each of the integrals (5') and (6') is an exact differential for all values of α and β and hence their derivatives with respect to α and β will be exact differentials. Differentiating (5′) with respect to α and β and equating to zero, we obtain the two exact differential equations

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* There is an example showing this in my thesis, "Anwendungen des Unabhängigkeitssatzes auf die Lösung der Differentialgleichungen der Variationsrechnung" (Göttingen 1906).

† *Math. Annalen*, vol. 58.

$$(7)\quad (y'-p)(F_{pp}p_\alpha + F_{pq}q_\alpha) + (z'-q)(F_{pq}p_\alpha + F_{qq}q_\alpha) = 0,$$

$$(8)\quad (y'-p)(F_{pp}p_\beta + F_{pq}q_\beta) + (z'-q)(F_{pq}p_\beta + F_{qq}q_\beta) = 0.$$

If B and C represent the coefficients of y' and z' in equation (7), then the remaining terms A are $-pB-qC$; in the same way, B_1, C_1 and A_1 representing corresponding members of equation (8), $A_1 = -pB_1 - qC_1$. The equations being exact,

$A_y = B_x$, $A_z = C_x$, $B_z = C_y$ and $A_{1y}=B_{1x}$, $A_{1z}=C_{1x}$, $B_{1z}=C_{1y}$.

The identity

$$(9)\quad \frac{\partial}{\partial x}\begin{vmatrix} B & B_1 \\ C & C_1 \end{vmatrix} + \frac{\partial}{\partial y}p\begin{vmatrix} B & B_1 \\ C & C_1 \end{vmatrix} + \frac{\partial}{\partial z}q\begin{vmatrix} B' & B_1' \\ C' & C_1' \end{vmatrix} \equiv 0$$

therefore follows. This method applied to integral (6′) gives the second identity

$$(10)\quad \frac{\partial}{\partial x}\begin{vmatrix} B' & B_1' \\ C' & C_1' \end{vmatrix} + \frac{\partial}{\partial y}p\begin{vmatrix} B' & B_1' \\ C' & C_1' \end{vmatrix} + \frac{\partial}{\partial z}q\begin{vmatrix} B & B_1 \\ C & C_1 \end{vmatrix} \equiv 0,$$

where B', B_1', C_1', C_1' have corresponding meanings.

If we denote by $m(x, y, z, p, q)$ the quotient

$$\begin{vmatrix} B & B_1 \\ C & C_1 \end{vmatrix} \div \begin{vmatrix} B' & B_1' \\ C' & C_1' \end{vmatrix}$$

we obtain for m the identity in x, y, z, α and β

$$(11)\quad \frac{\partial m}{\partial x} + p\frac{\partial m}{\partial y} + q\frac{\partial m}{\partial z} \equiv 0,$$

$$[p = p(x, y, z, \alpha, \beta), \qquad q = q(x, y, z, \alpha, \beta)].$$

$$m \equiv \begin{vmatrix} F_{pp} & F_{pq} \\ F_{pq} & F_{qq} \end{vmatrix} \div \begin{vmatrix} F'_{pp} & F'_{pq} \\ F'_{pq} & F'_{qq} \end{vmatrix}$$

is a known function of x, y, z, p and q.

$$y' = p(x, y, z, \alpha, \beta), \quad z' = q(x, y, z, \alpha, \beta)$$

is moreover according to the independence theorem an intermediary integral of the system of equations (4). If now we choose any particular solution of equations (4) $y = \bar{y}(x)$, $z = \bar{z}(x)$, then a pair of values of α and β always exists, which we will write α_0, β_0, such that $p(x, \bar{y}(x), \bar{z}(x), \alpha_0, \beta_0) \equiv \bar{y}(x)$, $q(x, \bar{y}(x), \bar{z}(x), \alpha_0, \beta_0) \equiv \bar{z}(x)$. When we substitute for y, z, p and q in the identity (11) the functions $\bar{y}(x)$, $\bar{z}(x)$, $\bar{y}'(x)$, $\bar{z}'(x)$, we obtain an identity in x; but (11) becomes in this case the total differential quotient of m with respect to x.

$$\therefore \frac{dm}{dx} = 0, \quad m = \text{const.}$$

Since $\bar{y}(x)$, $\bar{z}(x)$ were chosen arbitrarily, every solution of the system of equations (4) substituted in m gives a constant; hence

$$m(x, y, z, y', z') \equiv \begin{vmatrix} F_{y'y'} & F_{y'z'} \\ F_{z'y'} & F_{z'z'} \end{vmatrix} \div \begin{vmatrix} F'_{y'y'} & F'_{y'z'} \\ F'_{y'z'} & F'_{z'z'} \end{vmatrix} = \text{const.}$$

is an integral of the system (4).

A substitution which leaves system (4) invariant transforms the integral of which (4) are the Lagrange equations either into itself or into a new integral which has the same Lagrange equations. In this latter case, the one integral being given, the construction of such a substitution is equivalent to the construction of an integral.

CORNELL UNIVERSITY,
December, 1906.

ALGEBRAIC NUMBERS AND FORMS.

Zahlentheorie. Fünfter Teil: *Allgemeine Arithmetik der Zahlenkörper.* By PAUL BACHMANN. Leipzig, B. G. Teubner, 1905. xxii + 548 pp.

Einleitung in die allgemeine Theorie der Algebraischen Grössen. By JULIUS KÖNIG. Leipzig, B. G. Teubner, 1903. x + 564 pp.

THERE has been but little activity in America in this important and fascinating field. It seems appropriate, therefore, to preface this review with an elementary introduction to the subject. We shall consider the simpler features of the theory of quadratic number systems, for which the phenomena are typical of the general case, while the treatment may be made so simple that the fundamental ideas are not obscured by the algebraic intricacies and abstract character of the general theory. It is highly desirable that a very large circle of readers shall acquire a clear insight into the nature of this important field; it is hoped that not a few will be induced by this introduction to pursue this interesting subject in its generality, at least as far as developed in the admirable treatises under review.

Although the fundamental laws obeyed by integers, such as unique decomposition into primes, were observed by Gauss to hold true for his complex integers $a + bi$, this is rarely the case with the system of integral algebraic numbers determined by a root of a given algebraic equation with integral coefficients. Thanks to the genius of Kummer, Dedekind, and Kronecker, the introduction of "ideals" brought complete harmony out of chaos, and marked one of the greatest triumphs of mathematical endeavor. The elaboration and extension of the theory have given rise to an extensive literature, a detailed report of which has been prepared by Hilbert, *Jahresbericht der Deutschen Mathematiker-Vereinigung*, volume 4 (1894–95), pages 175–546. In addition to its great theoretical importance, the subject has vital relations with other branches of mathematics, *e. g.*, Galois's theory of algebraic equations, algebraic functions and their integrals, and diophantine equations.

A number τ is called algebraic if it is the root of an equation

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0,$$

with $a_1, \cdots, a_n$ rational numbers. Of all such equations satisfied by τ, there is an unique one of minimum degree m, necessarily irreducible, and τ is called an algebraic number of the mth degree. When $m = 2$, τ is called quadratic.

If the above coefficients a_i are integers, τ is called an integral algebraic number. Examples are $\sqrt{2}$, $3\sqrt{-1}$, $\frac{1}{2} + \frac{1}{2}\sqrt{-3}$.

We consider the system $R(\tau)$ of all rational functions of τ with rational coefficients, where τ is a given quadratic algebraic number. We may set $\tau = r + s\sqrt{d}$, r and s rational numbers, $s \neq 0$, while d is an integer, other than $+ 1$, not divisible by a perfect square. Evidently the system $R(\tau)$ is identical with the system $R(\sqrt{d})$ of all rational functions of $\sqrt{d}$ with rational coefficients.

Lemma I. In $R(\sqrt{d})$, the integral algebraic numbers are given by $x + y\theta$, where x and y are integers, and

$$\theta = \sqrt{d}, \text{ if } d \equiv 2 \text{ or } d \equiv 3 \pmod 4;$$

$$\theta = \tfrac{1}{2}(1 + \sqrt{d}), \text{ if } d \equiv 1 \pmod 4.$$

Consider $a + b\sqrt{d}$, a and b being rational, $b \neq 0$. It and its conjugate $a - b\sqrt{d}$ satisfy the equation

$$z^2 - 2az + a^2 - db^2 = 0.$$

Assuming that the coefficients are integers, we determine the character of a and b. Since $2a$ and $4a^2 - 4db^2$ are integers, while d has no square factor, $2b$ must be integral: Hence

$$a = \alpha/2,\ b = \beta/2, \qquad (\alpha \text{ and } \beta \text{ integers}).$$

It remains to require that $a^2 - db^2$, viz., $\frac{1}{4}(\alpha^2 - d\beta^2)$, be integral. If $d \equiv 2 \pmod 4$, α^2 must be even and hence α and β both even. If for $d \equiv 3 \pmod 4$ α were odd, $\frac{1}{4}(1 - 3\beta^2)$ could not be integral. If $d \equiv 1 \pmod 4$, α and β must both be even or both odd, so that $a + b\sqrt{d}$ is of the respective forms $\lambda + \mu\sqrt{d}$, $\lambda + \mu\sqrt{d} + \theta$, where λ and μ are integers and $\theta = \frac{1}{2}(1 + \sqrt{d})$. Setting $x = \lambda - \mu$, $y = 2\mu$ or $2\mu + 1$, respectively, we get $x + y\theta$. The lemma is therefore proved.

Given two integral algebraic numbers α and β of a domain* $R(\theta)$, we shall say that α is divisible by β if there exists in $R(\theta)$ an integral algebraic number q such that $\alpha = \beta q$.

For $d = -1$, the integral algebraic members of $R(i)$ are Gauss's complex integers $x + yi$. Two complex integers α and α_1 have a greatest common divisor δ, defined by the two properties:

1. α and α_1 are divisible by δ;
2. Every common divisor of α and α_1 is a divisor of δ.

Here δ may be determined (uniquely up to a factor $\pm 1, \pm i$) by the following process: Set $\alpha/\alpha_1 = a + bi$ and determine a complex integer $\xi = x + yi$ such that the norm $(a - x)^2 +$

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*Our interest will always center in a domain defined by a root θ of a given algebraic equation. There is little interest in the arithmetic of the system of all integral algebraic numbers of all degrees. In fact, decomposition is then unlimited. Thus $\alpha = \alpha^{\frac{1}{2}}\alpha^{\frac{1}{2}}$, where $\alpha^{\frac{1}{2}}$ is integral algebraic when α is.

$(b-y)^2$ of $\alpha/\alpha_1 - \xi$ shall be less than unity; we have only to select integers x and y such that $|a-x| \leqq \frac{1}{2}$, $|b-y| \leqq \frac{1}{2}$. Then

$$\alpha = \xi\alpha_1 + \alpha_2, \quad \text{norm}\,\alpha_2 < \text{norm}\,\alpha_1,$$

where α_2 is a complex integer. If $\alpha_2 \neq 0$, we proceed similarly with α_1 and α_2, and determine complex integers ξ_1 and α_3 such that

$$\alpha_1 = \xi_1\alpha_2 + \alpha_3, \quad \text{norm}\,\alpha_3 < \text{norm}\,\alpha_2.$$

Since the norms of $\alpha_1, \alpha_2, \alpha_3, \cdots$ form a series of decreasing positive integers, we must reach a term α_{r+1} of zero norm. Then α_r is the required number δ. It is now a simple matter to prove* that every complex integer can be expressed as a product of primes in one way and essentially but one way (*i. e.*, up to a factor ± 1, $\pm i$).

In general, the state of affairs is entirely different. For illustration, we take Dedekind's simple example $d = -5$ (Dedekind, l. c., page 451 and 547; König, pages 19, 93). In $R(\theta)$, where $\theta = \sqrt{-5}$, the integral algebraic numbers are $x + y\theta$, x and y integers (Lemma I). Here there are two ways of factoring 9, viz.,

$$(1) \qquad 3 \cdot 3 = (2+\theta)(2-\theta),$$

while 3, $2 \pm \theta$ differ from the units (here ± 1) and each is indecomposable in $R(\theta)$, *i. e.*, has no factor other than itself, its negative, and ± 1. For example, if

$$2 + \theta = (x + y\theta)(z + w\theta),$$

then

$$2 - \theta = (x - y\theta)(z - w\theta).$$

By multiplication (or by taking the norms in the first equation),

$$(2) \qquad 9 = (x^2 + 5y^2)(z^2 + 5w^2).$$

But $x^2 + 5y^2 = 3$ is not solvable in integers. Hence one of the assumed factors of $2 + \theta$ must be ± 1. This discussion of

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* Dirichlet-Dedekind, Zahlentheorie, 1894, pp. 434–450 On p 450 Dedekind cites eight further examples of quadratic number systems in which an analogous g. c. d. process holds. For the case of negative discriminant, Birkhoff has determined geometrically (*Amer. Math. Monthly*, Aug , 1906) all such quadratic number systems.

(2) shows that also 3 is indecomposable. Hence 9 has two sets of indecomposable factors (1). There are other respects in which the laws of arithmetic here fail. Although 3 is indecomposable it does not have the true nature of a prime, since by (1) 3 divides the product $(2+\theta)(2-\theta)$ but does not divide either factor. Again as König points out, the numbers 9 and $3-6\theta$ have no greatest common divisor in the sense 1, 2, above. In fact the only factors (apart from sign) of 9 are $1, 3, 9, 2 \pm \theta$; the only factors of $3-6\theta$ are $1, 3, 1-2\theta, 2-\theta, 4-\theta, 3-6\theta$. The common factors are $1, 3, 2-\theta$, no one of which is divisible by the other two, as seen above.

To overcome these difficulties Kummer would introduce "ideal prime numbers" α, β, γ, such that

$$3=\alpha\beta, \quad 2+\theta=\alpha^2, \quad 2-\theta=\beta^2, \quad 1-2\theta=\beta\gamma.$$

Thus each member of (1) decomposes further into $\alpha^2\beta^2$, so that there is only one decomposition of 9 into ideal primes. Again, there is now a greatest common divisor of 9 and $3-6\theta=\alpha\beta^2\gamma$, viz., $\alpha\beta^2$. Dedekind* has given a complete treatment by Kummer's ideals of the laws of divisibility in the system of integral algebraic numbers of this domain $R(\sqrt{-5})$; he emphasizes, however, the delicacy of the problem and the necessity of "the greatest circumspection." Kummer and his followers succeeded in applying his method to but few types of domains (cf. Bachmann, pages 150–159). In addition to the practical difficulties, there is the logical objection to Kummer's theory that an ideal number is not defined in itself, but merely its presence or absence as a factor of an existing integral complex number (the criteria being congruential conditions). As Kummer's work is now of mere historical interest, we pass to Dedekind's method which is open to none of the objections cited. We shall go into details only for quadratic domains.

Let θ denote a fixed quadratic number defined as in Lemma I, and consider the system $I(\theta)$ of integral algebraic numbers $\xi=x+y\theta$, x and y being integers. If μ is a fixed number of $I(\theta)$, the set of products $\xi\mu$ is closed under addition and subtraction, since $\xi_1\mu \pm \xi_2\mu=(\xi_1 \pm \xi_2)\mu$; and also under multiplication by any number ρ of I, since $\rho(\xi\mu)=(\rho\xi)\mu$, and since the product $\rho\xi$ of two numbers of I belongs to I. This particular set of all the multiples of μ will be called a principal ideal (μ). In general, we define as an ideal any system S of numbers * of $I(\theta)$ which have the two properties stated, viz.,

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* *Bulletin des Sciences mathématiques et astr.*, ser. 2, vol. 1 (1877), p. 69. In a series of five articles in this and the preceding volume Dedekind gives an elementary account of Kummer's method and the origin of his own theory.

(A) The sum and difference of any two (equal or distinct) numbers of the system S are themselves members of this system S;

(B) Every product of a number of the system S and a number of the system $I(\theta)$ is a number of the system S.

We proceed to give a simple formula for the numbers of such an ideal S. We first investigate systems S having property (A) only. The numbers of S fall into two sets: $k_1, k_2, k_3, \cdots$, and $l_1 + m_1\theta,\ l_2 + m_2\theta, \cdots$, where no $m_i = 0$. Let k be the greatest common divisor of the integers $|k_1|, |k_2|, \cdots$, and m that of $|m_1|, |m_2|, \cdots$ A suitably chosen linear combination of the $l_i + m_i\theta$ gives $l + m\theta$, l an integer. Hence the system S contains k and $l + m\theta$. Conversely, any number $x + y\theta$ of S is a linear function of k and $l + m\theta$. For if $y = 0$, x must be a multiple of k; while if $y \neq 0$, then $y = qm$, so that

$$(x + y\theta) - q(l + m\theta) \equiv x - ql$$

is an integer and hence of the form $q'k$. We thus have

Lemma II. Any system of numbers of $I(\theta)$ which has property (A) may be exhibited as the set $[k, l + m\theta]$ of all linear homogeneous functions with integral coefficients of k and $l + m\theta$, where k and m are positive integers and l an integer.

We next require that S shall have also property (B). The necessary and sufficient conditions are that $k\theta$ and $(l + m\theta)\theta$ shall belong to S. By Lemma I, there are two cases

$$(3) \qquad \theta^2 - d = 0\,; \quad \theta^2 - \theta + \tfrac{1}{4}(1 - d) = 0.$$

In either case, the preceding conditions require that k and l be divisible by m. Hence $S = [ma, m(b + \theta)]$, where a and b are integers, $a > 0$. In this notation, we examine more minutely the condition that $m(b + \theta)\cdot\theta$ shall belong to S. For case (3_1), this equals $md + mb\theta$. But $b \cdot m(b + \theta)$ occurs in S. By subtraction, we get the integer $mb^2 - md$, which occurs in S if and only if $b^2 - d$ is a multiple of a. Treating case (3_2) similarly, we obtain

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* We exclude the system composed of zero only.

Lemma III. For a quadratic domain defined by a root of one of the equations (3), the ideals are given by $[ma, m(b+\theta)]$, where a, b and m are integers subject to the respective conditions

$$(3') \quad b^2 - d \equiv 0 \pmod{a}; \quad b^2 + b + \tfrac{1}{4}(1-d) \equiv 0 \pmod{a}.$$

Multiplication of ideals is defined as follows: If μ ranges over the numbers of an ideal S, and μ' over the numbers of an ideal S', then the products $\mu\mu'$ and their sums form an ideal S'', called the product of the factors S, S', and designated SS'.

In particular, for $\theta^2 = d$, the product of

$$S = [ma, m(b+\theta)], \ b^2 \equiv d \pmod{a},$$

and the conjugate ideal

$$S_1 = [ma, m(-b+\theta)]$$

is the ideal SS_1, expressed initially in the form

$$[m^2a^2, m^2a(-b+\theta), m^2a(b+\theta), m^2(-b^2+d)],$$

viz., the aggregate of the linear homogeneous functions with integral cofficients of these four numbers. Let c denote the integer $(b^2-d)/a$. After obvious modifications, we have

$$SS_1 = [m^2a^2, 2m^2ab, m^2ac, m^2a(b+\theta)].$$

The greatest common divisor of the first three numbers is m^2ag, where g denotes that of a, $2b$, c. If a and c had the common factor 2, then $b^2 - d = ac \equiv 0 \pmod{4}$, and $b^2 \not\equiv 0 \pmod{4}$ since d has no square factor; hence would $b^2 \equiv 1$, $d \equiv 1 \pmod{4}$, which contradicts the present hypothesis (Lemma I). Hence g is odd. If $g > 1$, $b^2 - d = ac \equiv 0 \pmod{g^2}$, and d would have a square factor g^2. Hence $g = 1$. Since a, $2b$, c have the greatest common divisor 1, a suitable linear combination of them equals 1. Hence

$$SS_1 = [m^2a, m^2a(b+\theta)] = [m^2a, m^2a\theta],$$

and hence is formed of all the complex integral multiples of m^2a, *i. e.*, is the principal ideal (m^2a).

For the case (3_2), we denote the second number $(3')$ by ac, and find similarly that the greatest common divisor of a, $2b+1$, c is unity, since $d=(2b+1)^2-4ac$. Setting $S_1=[ma, m(b+\bar{\theta})]$, where $\bar{\theta}$ is the second root of (3_2), we find as before that SS_1 is the principal ideal (m^2a). Hence we have

Theorem I. For a quadratic domain, the product of any ideal and its conjugate is a principal ideal.

Corollary. If $SS'=SS''$, then $S'=S''$.

For, if S_1 is the ideal conjugate to S, then SS_1 is a principal ideal, say (t), where t is an integer. Then

$$S_1SS'=S_1SS'',\ (t)S'=(t)S'',$$

so that the ideals S' and S'' include the same numbers.

Theorem II. If all the numbers of an ideal C belong to an ideal A, there exists an ideal B such that $AB=C$, and conversely.

Under this first hypothesis, the numbers of CA_1 belong to the principal ideal $AA_1=(t)$, A_1 being the conjugate to A. Thus the numbers of CA_1 are $\lambda_1t, \lambda_2t, \cdots$, the λ's in $I(\theta)$. Since properties (A) and (B) hold for the ideal CA_1, we have, for every number ρ of $I(\theta)$,

$$\lambda_1t+\lambda_2t=\lambda_it,\ \lambda_1t-\lambda_2t=\lambda_jt,\ \rho(\lambda_1t)=\lambda_kt,$$

λ_i, λ_j, λ_k belonging to the set $\lambda_1, \lambda_2, \cdots$. Since the factor t may be dropped, the numbers $\lambda_1, \lambda_2, \cdots$ themselves have the characteristic properties (A) and (B) of an ideal, and hence form an ideal B. From $CA_1=B(t)$ and $AA_1=(t)$ follows $C=AB$, by the preceding corollary.

The converse proposition that every number of AB belongs to A follows from the definition of multiplication of ideals and properties (B), (A).

According to Dedekind, who is followed by Bachmann, an ideal C is said to be divisible by an ideal A if all the numbers of C belong to (occur in) A. According to Hurwitz,* Hilbert, and others, an ideal C is divisible by an ideal A if there exists an ideal B such that $C=AB$. In Theorem II, we have shown (for quadratic domains) that the two definitions † are in complete accord.

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* "Ueber die Theorie der Ideale," *Göttinger Nachrichten*, 1894, pp 291–298.

† Dedekind's objections to the second definition are given in the *Göttinger Nachrichten*, 1895, pp. 106–113.

Lemma IV. A positive integer t occurs only in a finite number of ideals of a given quadratic number domain.

For, if $[k, l + m\theta]$ is an ideal containing t, it follows from the proofs of Lemmas II and III, that k is a divisor of t, and m a divisor of k, while obviously l can be reduced modulo k.

Theorem III. Any ideal A is divisible by only a finite number of ideals.

If A_1 is the ideal conjugate to A, then AA_1 is a principal ideal (t), by Theorem I. This integer t thus occurs in every ideal which divides A.

The principal ideal (1), which is evidently composed of all the integral algebraic numbers of the domain, plays the rôle ot unity in multiplication and division. An ideal, different from (1) and divisible by no ideal other than itself and (1), is called a prime ideal.

Theorem IV. If the prime ideal P divides the product AB, it divides A or B.

Suppose that P does not divide A. Then the ideal composed of the linear combinations of the numbers of both A and P divides P and yet is distinct from P, and hence is (1). Hence $1 = \alpha + \pi$, where α is some number of A, π some number of P. Let β be any number of B. Then $\beta = \alpha\beta + \pi\beta$. By hypothesis, $\alpha\beta$ occurs in P. Hence by the definition of ideals, β occurs in P. Hence B is divisible by P.

Theorem V. Every ideal A, other than (1), can be expressed in one and but one way as a product of a finite number of prime ideals.

If A is not itself a prime ideal, it has a divisor A_1 distinct from (1) and A, and $A = A_1A_2$, where A_2 is distinct from A and (1). If one of the ideals A_1 and A_2 is not a prime ideal, it equals the product of two ideals, and we get $A = A_1'A_2'A_3'$. This procedure ultimately terminates, so that A is divisible by a prime ideal. In fact, by Theorem III, A is divisible by only a finite number n of ideals. Hence A is not equal to a product of more than n ideals, equal or distinct, but $\neq (1)$, since a relation $A = B_1B_2 \cdots B_{n+1}$ would require the existence of the $n + 1$ distinct ideal divisors $B_1, B_1B_2, \cdots, B_1B_2 \cdots B_{n+1}$. We therefore obtain a factorization $A = P_1 \cdots P_r$ into prime ideals.

This factorization is unique in view of Theorem IV.

It now follows that ideals obey the fundamental laws of divisibility holding for integers. The arithmetic of ideals thus becomes a subject of decided interest and importance.

There remains the question of the disposition to be made of the difficulties encountered in the arithmetic of the integral algebraic numbers forming the system $I(\theta)$. To every number μ of $I(\theta)$ corresponds a principal ideal (μ), and conversely. If $\mu\lambda = \nu$, then $(\mu)(\lambda) = (\nu)$. To each non-principal ideal A we may make correspond * a fictitious entity called an "ideal number" α, such that $AB = C$ implies $\alpha\beta = \gamma$. When A is a prime ideal, α is called a "prime ideal number." This scheme is not to be confused with Kummer's. He required very complicated machinery which worked only for special domains. On the contrary, we are now throwing the burden on the simple theory of ideals and deriving by formal correspondence the needed properties of the ideal numbers.

The text by Bachmann is very appropriately dedicated to Dedekind, as it develops Dedekind's original theory of integral algebraic numbers and employs almost exclusively the methods insisted upon by the latter. In a few instances, however, Bachmann departs from Dedekind's purely arithmetical standpoint which does not permit the use of arbitrary variables and undetermined coefficients. The deviations occur in the exposition of Hensel's work and in making use of the simplifications due to Hurwitz and others. In fact, König (page 482) insists that "die von verschiedenen Autoren für die Dedekind'sche Idealtheorie gegebenen 'Vereinfachungen' beruhen durchweg auf einer mehr oder weniger verhüllten Anwendung der Kroneckerschen Grundideen." The exposition by Dedekind (Zahlentheorie, pages 434–657) is in parts very elementary and amply illustrated by simple examples, but in other parts is very abstract, requiring the reader to hold in mind an array of technical concepts, symbols and names. By using fewer abstract proofs and adopting a more expansive style of presentation, Bachmann has produced a book everywhere readable. As noted above for ideals, so for the more general concept modulus, viz., a system of numbers closed under addition and subtraction, Dedekind says that if all the numbers of a modulus C occur also in the modulus A then C is divisible by A

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* Instead of relying upon the principle of correspondence, it seems allowable to the reviewer to define our ideal "numbers" to be the ideals themselves. In thus speaking of an ideal (*viz.*, a certain aggregate of integral algebraic numbers) as a "number," we have the precedent that certain ordered aggregates $(a_1, a_2, \cdots, a_n)$ of real numbers $a_1, \cdots, a_n$ are called hypercomplex numbers, the customary limitation that n shall be finite not being essential. Again, we may define an irrational number to be the inferior class of rational numbers obtained by a Dedekind cut.

and C is a multiple of A, and writes $C > A$. Since the aggregate C is smaller than the aggregate A, the terms "divisible" and "multiple" are used in a technical sense, reverse to the usual numerical sense. While one may carry in mind the technical names it is asking too much to reason with a familiar symbol in the reverse of customary usage. We therefore welcome Bachmann's introduction of an entirely new symbol for "contained in" and replacing Dedekind's $>$. It would seem preferable, however, to the reviewer to use the symbol $a \equiv 0\ (A)$ to denote that the number a is contained in the modulus A, and similarly $C \equiv 0\ (A)$. In the spirit of modular systems, this notation would naturally mean that the numbers of C are contained in the modulus A, and also, as in the elements of congruences, that C is divisible by A. If Dedekind's theory of the modulus were presented with this notation, the reader would not find it necessary continually to struggle with himself. For the special case of ideals, the notation is used by Hilbert in his Report (page 183).

Bachmann states at the bottom of page 154 that for quadratic domains $R(\sqrt{d})$ the introduction of ideals is necessary if there is more than one class of quadratic forms of determinant d. There may however be two classes differing by the factor -1 (cf. Dedekind, l. c., page 451). A few misprints may be noted; page 227, line 7, contains a misprint for p; page 231, line 12, contains a misprint for γ.

The present volume by Bachmann serves admirably its purpose of affording a simple and attractive introduction to Dedekind's theory of the general arithmetic of algebraic numbers; there is promised a supplementary volume treating of special types of number domains.

The treatise by König is much more than a presentation of results contained in the memoirs of Kronecker and his followers; it must be regarded also in the light of an original contribution to the subject, containing new points of view, completing fragmentary results, replacing incomplete by adequate proofs, and undertaking new developments of fundamental nature. The text is far more than a commentary on Kronecker's fundamental "Festschrift zu Herrn E. E. Kummer's Doktor-Jubiläum, 10. Sept. 1881." The subjects treated include the divisibility of forms, factorization of forms, Kronecker's abstract formulation of the adjunction of a root of an algebraic equation, Galois's theory, Kronecker's method of elimination, general theory of resultants, discriminants, functional determinants, algebraic manifolds, divisor systems (modular systems), algebraic and arithmetic theories of linear diophantine problems, theorems of Noether and Hilbert, and finally the theory of integral algebraic quantities. The concepts of this rich array of general material are so interwoven that a report upon a particular part would be entirely unsatisfactory, while the limitations of space here preclude a survey of the whole. Although the theory of ideals does not play as predominating a rôle in this subject as in the theory of algebraic numbers, a few remarks in this direction will afford a suitable sequel to the earlier part of this composite review. The introduction of "ideal quantities" to bring harmony into the laws of divisibility of integral algebraic forms is accomplished in a most pleasing manner by König.

At the outset we consider the two domains

$$(A, x_1, \cdots, x_m), \qquad [A, x_1, \cdots, x_m],$$

formed respectively of all rational, and all rational integral functions of the indeterminates $x_1, \cdots, x_m$ with coefficients in A, where (for the purely arithmetical theory) A denotes the system $[1]$ of all integers, while (for the algebraic theory) A denotes a field* composed of real or complex numbers (some of which may be indeterminates other than the x's). Let $F_0, \cdots, F_n$ denote any forms in the domain $[A, x_\iota]$ such that

$$F_0 z^n + F_1 z^{n-1} + \cdots + F_n = 0$$

is irreducible in (A, x_ι). Let its roots be $\alpha_1, \cdots, \alpha_n$. Let

$$\Gamma = (A, x_1, \cdots, x_m\,;\, \alpha)$$

be the domain obtained by the adjunction of $\alpha = \alpha_1$ to (A, x_ι). Then Γ is formed of the quantities

$$g(\alpha) = (G_0 + G_1\alpha + \cdots + G_{n-1}\alpha^{n-1}) \div H,$$

where the G's and H are any forms in $[A, x_\iota]$ such that $H \neq 0$. Then

$$\text{norm } g(\alpha) \equiv \prod_{\iota=1}^{n} g(\alpha_\iota)$$

equals the quotient of two quantities of $[A, x_\iota]$.

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* An aggregate closed under the four rational operations, addition, etc.

Let $u_1, \cdots, u_n$ be new indeterminates and let $U_1, U_2, \cdots$ denote distinct products of powers of the u's subject only to the condition that $U_r = U_s$ implies $r = s$. We consider the general form

$$f(\alpha) \equiv \sum g_r(\alpha) U_r.$$

It satisfies the equation

$$\text{norm } (z - f) \equiv \prod_{i=1}^{n} [z - f(\alpha_i)] = 0$$

whose coefficients are forms in $u_1, \cdots, u_n$, the coefficients of these forms being quotients of quantities in $[A, x_i]$ and therefore quantities in (A, x_i). This norm is an exact power of a form $N(z)$, irreducible in (A, x_i, u_i) *, as follows readily from the irreducibility of the equation for the α's. We may suppose the coefficients in $N(z) = 0$ free of denominators. It is a fundamental theorem in Kronecker's theory that there exists a greatest common divisor process for two forms in a domain $[A, x_i, u_i]$, when there is such a process for numbers of A; the latter is obviously here the case since A is either a field or the system [1] of all integers. Hence our form f satisfies an equation $G_0 z^d + \cdots + G_d = 0$, irreducible in $[A, x_i, u_i]$, such that $G_0, \cdots, G_d$ have no common divisor. In case G_0 is a unit (divisor of 1), we may set $G_0 = 1$ and call the root f an "integral algebraic form" (with respect to the domain Γ). It is easy to establish the equivalence of this definition with the following. The form f is an integral algebraic form if and only if it satisfies some equation

$$z^d + G_1 z^{d-1} + \cdots + G_d = 0$$

whose coefficients are forms of the domain $[A, x_i, u_i]$. By suppressing the indeterminates u_i we obtain the definition of the integral algebraic quantities of Γ. Denote by $[\Gamma]$ and $[\Gamma]_u$ the domains composed of all these integral algebraic quantities and forms, respectively. Each domain is closed under addition, subtraction and multiplication.

A non-field domain D is called complete (vollständig) when any two of its quantities have in D a greatest common divisor δ, in the sense 1, 2, above. As already noted, $[A, x_i]$ is a complete domain. But a domain $[\Gamma]$, just defined, is complete

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* The u's should appear in the notations of the domains, bottom of p. 464 (König).

only in the very simplest cases; this point is illustrated in the above example of the numbers $a + b\sqrt{-5}$, a and b integers. Hence we cannot here define, as in algebra, a primitive form to be one in which the greatest common divisor of its coefficients is 1 (or a unit). Following Kronecker (Festschrift, § 15), we say that for a domain $[\Gamma]$ a form ϕ is "primitive" if norm ϕ is a primitive form of the indeterminates u_i in the elementary sense.

We may now make a clear statement of the problem of the association of ideal quantities. We seek to enlarge the domain $[\Gamma]$ into a complete domain $[G]$ in such a manner that not merely the additive and multiplicative combinations, but also the properties relating to divisibility of the quantities of $[\Gamma]$ shall remain valid in the new domain $[G]$. In symbols, if α, β, γ are any quantities in $[\Gamma]$, there shall exist quantities α', β', γ' in $[G]$ such that $\alpha + \beta = \gamma$ implies $\alpha' + \beta' = \gamma'$, $\alpha\beta = \gamma$ implies $\alpha'\beta' = \gamma'$, and conversely, while if α is [or is not] divisible by β then α' is [or is not] divisible by β', and conversely. The most exacting requirement is that $[G]$ shall be a complete domain.

All these conditions are met in the simplest manner by the domain $[G]$ of all the quotients γ_u/ϵ_v, where γ_u is an arbitrary form of $[\Gamma]_w$ and ϵ_v an arbitrary primitive form of $[\Gamma]_w$, addition and multiplication in $[G]$ being defined by

$$\frac{\gamma_u}{\epsilon_v} + \frac{\gamma'_{u'}}{\epsilon'_{v'}} = \frac{\gamma_u\epsilon'_{v'} + \gamma'_{u'}\epsilon_v}{\epsilon_v\epsilon'_{v'}}, \quad \frac{\gamma_u}{\epsilon_v} \cdot \frac{\gamma'_{u'}}{\epsilon'_{v'}} = \frac{\gamma_u\gamma'_{u'}}{\epsilon_v\epsilon'_{v'}}.$$

The sum and product belong to $[G]$ since the product of two primitive forms ϵ_v and $\epsilon'_{v'}$ is a primitive form, and since the domain $[\Gamma]_w$ is closed under addition and multiplication. The units of $[G]$ are the quantities ϵ'_u/ϵ_v, where also the numerator is primitive. One of the conditions that $[G]$ be a complete domain is that it be not a field,* and this is satisfied since $2x = 1$ is not solvable in $[G]$. In fact, $2\gamma_u = \epsilon_v$ is impossible since the norm of the primitive form ϵ_v is not divisible by 2. Finally, any two quantities γ_u/ϵ_v and $\gamma'_{u'}/\epsilon'_{v'}$ of $[G]$ have in $[G]$ a greatest common divisor, which (or any product of it by a unit) may be exhibited as the form

$$\delta_w = \sum_{(r)} C_r W_r + \sum_{(s)} C_s W_s,$$

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* Were it not for this condition, we could take as $[G]$ simply the field $[\Gamma]$ composed of all quotients of the quantities of $[\Gamma]$.

where C_1, C_2, $\cdots$ denote the coefficients of the form γ_w; C'_1, C'_2, $\cdots$ the coefficients of $\gamma'_{w'}$; while W_r and W'_s denote distinct products of powers of the variables $w_1, w_2, \cdots$. Naturally, we do not enter here upon the proof* that δ_w has the properties 1, 2 of the greatest common divisor.

The quantities of $[G]$ are called "ideal quantities" of $[\Gamma]$, although some of them already occur in $[\Gamma]$. That the ideals here admit of addition as well as multiplication is a pleasing feature of Kōnig's theory not found in Dedekind's theory of ideals, nor in Kronecker's exposition. Ideals of the Kōnig type appear in Weber's treatment (Algebra, volume 2) of the more special theory of algebraic numbers.

L. E. Dickson.

The University of Chicago.

CORRECTION.

Professor Wilczynski has kindly called my attention to the following misstatements in my review of his Projective Differential Geometry (Bulletin, pages 190–194).

P. 191. The second member of the expression for $\theta_{3,1}$ should be completed by adding $-27P_2\theta_3^2$.

P. 191, line 5 from bottom. Strike out first sentence and substitute "The osculating cubic may hyperosculate C_y."

P. 191. For last clause read: "If $\theta_{3,1} = 0$, then C_y is a curve of coincidence points."

P. 193, line 6. For θ_4 read θ_9.

P. 193, line 6 from bottom. For "second" read "same."

P. 193, line 2 from bottom. After "range" insert "in certain cases."

Virgil Snyder.

NOTES.

At the meeting of the London mathematical society held on February 14, the following papers were read: By G. A. Miller, "Groups defined by the order of the generators and the order of their commutator"; by T. Stuart, "On the reduction of the factorization of binary septans and octans to the solution of a Pellian"; by E. W. HOBSON, "On the repeated integrals"; by W. BURNSIDE, "The construction of the line drawn through a given point to meet two given lines."

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* Kōnig, p. 479. Misprints in the accents occur in the formula at the middle of this page; while at the top of p. 474, β_w should read β_s.

THE Rochester section of the association of teachers of mathematics in the middle states and Maryland was organized February 23, 1907, at the University of Rochester. After addresses by President R. Rhees, Professor D. E. Smith, Mr. A. H. Wilcox and Mr. H. S. West, the following papers were read: By Professor A. S. GALE, "The literature of secondary mathematics"; by Mr. W. BETZ, "Recent tendencies and open questions in the teaching of elementary geometry"; by Professor H. D. MINCHIN, "How the study of physics helps the teacher of mathematics."

THE fourth international congress of mathematicians will be held at Rome from April 6 to 11, 1908. The present plan is to emphasize the encyclopedic side of the science, to summarize the results already accomplished and to define sharply the great problems still unsolved. A series of formal addresses has been provided for, in addition to papers of less general interest. The following have consented to present addresses before the joint sessions of all the sections: Professors G. DARBOUX, A. R. FORSYTH, D. HILBERT, F. KLEIN, H. A. LORENTZ, G. MITTAG-LEFFLER, S. NEWCOMB, E. PICARD, H. POINCARÉ. For the more technical papers, the congress will be divided into four sections: (1) arithmetic, algebra and analysis, (2) geometry, (3) mechanics, mathematical physics, various applications of mathematics, (4) philosophical, historical and didactic questions.

A NEW mathematical review, the *Gazeto Matematika Internacia*, is projected, to be published in Esperanto. The *Gazeto* is not intended to compete with existing journals, but aims rather to establish a closer relation among them. It will publish articles on pure and applied mathematics, mechanics, theoretical physics, pedagogy, problems, book reviews, biographies, etc. The price is fixed for the present at $2.40 per annum, and the first volume will contain about 192 pages. All correspondence should be addressed to F. J. Vaes, Mathenesserlaan 290, Rotterdam.

PROFESSOR C. ALASIA, mathematical editor of the *Rivista di Fisica e Matematica*, will review in that journal all new publications sent to him at Ozieri, Italy.

The following university courses in mathematics are announced for the summer semester of 1907 :

UNIVERSITY OF INNSBRUCK. — By Professor K. TUMLIRZ: Mechanics, II, four hours; Capillarity, two hours; Seminar, two hours. — By Professor J. GMEINER: Algebra, II, four hours ; Double integrals, four hours; Seminar, two hours.—By Professor K. ZINDLER: Differential equations, four hours; Seminar, two hours. — By Professor J. MENGER: Projective geometry, II, four hours.

UNIVERSITY OF JENA. — By Professor R. HAUSSNER: Differential calculus, five hours ; Proseminar on plane analytic geometry, two hours ; Seminar, one hour; Selected chapters of geometry, two hours; Differential geometry, four hours. — By Professor J. THOMAE: Plane analytic geometry, four hours; Map-drawing, two hours. — By Professor G. FREGE : Theory of forces and the Newtonian law, four hours.

UNIVERSITY OF MUNICH. — By Professor F. LINDEMANN : Theory of elliptic functions, five hours ; Theory of algebraic forms, four hours; Seminar, two hours. — By Professor A. VOSS : Introduction to the theory of partial differential equations, four hours; Algebra, II, four hours; Seminar, two hours. — By Professor A. PRINGSHEIM: Integral calculus, five hours ; with geometric applications, two hours. — By Professor A. SOMMERFELD : Theory of radiation, three hours ; Foundations of thermodynamics, two hours; Seminar, two hours. — By Professor K. DOEHLEMANN: Descriptive geometry, II, three hours ; with exercises, two hours ; Geometric mechanics, three hours; with exercises, one hour. — By Professor E. v. WEBER : Analytic geometry of space, four hours ; with exercises, two hours; Differential calculus, four hours; with exercises, two hours. — By Dr. F. HARTOGS: Elementary geometry of two and three dimensions, three hours. — By Dr. O. PERRON : Analytic theory of numbers, four hours.

THE detailed scheme which embodies the resolutions passed by the senate of the University of Cambridge last year (BULLETIN, volume 13, page 203) has been finally adopted (February 2, 1907) by a vote of about 780 to 640. The new regulations come into force in 1910. The first senior wrangler was announced in 1747, and the last one will be recorded in the summer of 1909.

THE Adams prize has been awarded to Professor E. W. BROWN, for his essay on "The irregularities in the moon's motion due to the direct action of the planets." This prize was founded about sixty years ago to commemorate the work of J. C. Adams in connection with the discovery of Neptune. It is offered every two years for competition on a set subject and is open to anyone who has at any time taken a degree in the University of Cambridge. Among former prize winners are J. C. Maxwell, E. J. Routh, J. J. Thomson and J. H. Poynting. An award has been made on an average only about once in six years.

PROFESSOR M. D'OCAGNE, of the Ecole des Ponts et Chaussées, of Paris, will deliver during the months of March and April a course of lectures at the University of Paris on graphic calculus and nomography. In the latter part of the course he will give a complete account of his own methods in the subject.

AT the close of the present academic year Professor ULISSE DINI will have completed the fortieth year of his professorship at the University of Pisa. To mark the occasion, the faculty of sciences of the university proposes the presentation of a suitable memorial. Subscriptions to the fund should be sent to Professor Onorato Nicoletti, via S. Martino 3, Pisa, Italy.

PROFESSOR E. LAMPE, of the technical school at Charlottenburg, has been decorated with the order of the red eagle of the third class. Professor T. REYE, of the University of Strassburg, has received the order of the crown of the second class.

Mr. W. H. JACKSON, lecturer in mathematics at the University of Manchester, England, has been appointed associate professor of mathematics at Haverford College, Pennsylvania.

AT the College of the City of New York, Dr. COHEN has been promoted to an instructorship in mathematics, and Mr. G. H. HAYES has been appointed tutor in mathematics.

AT the University of Washington five teaching fellowships in mathematics have been established, with a compensation of of $400 to $500.

THE deaths are announced of F. P. H. STIRLING, professor of mathematics in the Christian College at Madras, and J. RAJEWSKI, associate professor of mathematics at Lemberg.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Abhandlungen über die regelmässigen Sternkörper. Abhandlungen von L. Poinsot (1809), A. L. Cauchy (1811), J. Bertrand (1858), A. Cayley (1859). Uebersetzt und herausgegeben von R. Haussner. Leipzig, Engelmann, 1906. 12mo. 129 pp. (Ostwald's Klassiker der exakten Wissenschaften) Cloth. M. 2.80

Carslaw (H. S.). Introduction to the theory of Fourier's series and integrals, and the mathematical theory of the conduction of heat. New York, Macmillan, 1907. 8vo. 17 + 434 pp. Cloth. $4.50

Eichler (K.). Beitrag zur Grassmannschen Punktrechnung. Hamburg, 1905. 8vo. 24 pp.

Glenn (O. E.). Determination of the abstract groups of order p^2qr; p, q, r being distinct primes. (Diss., University of Pennsylvania.) Lancaster, New Era Printing Co., 1906. 4to 14 pp. (*Transactions of the American Mathematical Society*, Vol. 7, pp. 137–151.)

Ilnicki (E.). Ueber die Prinzipien der Infinitesimalrechnung und über die Wandlungen, welche die Darstellung dieses Zweiges der Mathematik im Laufe seiner Entwicklung erfahren hat. (Progr.) Czernowitz, 1906. 8vo. 43 pp.

Laffitte (P. de). Essai sur le carré magique de N à N nombres. Paris, Gauthier-Villars, 1906. 8vo. 23 pp. F. 1.50

Myller (A.). Gewöhnliche Differentialgleichungen höherer Ordnung in ihrer Beziehung zu den Integralgleichungen. (Diss.) Göttingen, 1906. 8vo. 35 pp.

Schubert (H.). Mathematische Mussestunden. Eine Sammlung von Geduldspielen, Kunststucken und Unterhaltungsaufgaben mathematischer Natur. Grosse Ausgabe 3te Auflage. Vol. I. Zahl-Probleme. Leipzig, Göschen, 1907. 12mo. 8 + 200 pp. Cloth. M. 4.00

Smith (A. W.). The symbolic treatment of differential geometry. (Diss., Ogden Graduate School of Science.) Chicago, University of Chicago Press, 1907. 4to. 61 pp. $0.35

Wek (J.). Ueber die Steinerschen Ellipsen des Dreieckes. (Progr.) Marisch-Schönberg, 1906. 8vo. 20 pp.

Zoel (G.). Tratado de análisis matemática. Vol. III. Applicación del calculo infinitesimal al estudio de las figuras planas. Zaragoza, Casañal, 1905. 8vo. 320 pp. P. 7.00

II. ELEMENTARY MATHEMATICS.

Beaven (H. C.). Solutions of the examples in Hall's Easy graphs. London, Macmillan, 1907. 12mo. Cloth. 3s. 6d.

Bourlet (C.). Cours abrégé de géométrie. Vol. I. Géométrie plane. Paris, Hachette, 1907. 404 pp. Fr. 2.50

COLLAR (G.). New Algebra. As far as the binomial theorem, including chapters on graphs. London, Meiklejohn, 1907. 12mo. 436 pp. Cloth. 4s. 6d.

DASSEN (C. C.). Tratado elemental de geometria Euclidea. Vol. II. Geometria del espacio. Buenos Ayres, Hermanos, 1905. 12mo. 17 + 470 pp.

HENRICI und TREUTLEIN (P.). Lehrbuch der Elementar-Geometrie. Teil II: Aehnliche und perspektive Abbildung in der Ebene (Kegelschnitte), Berechnungen der ebenen Geometrie (Trigonometrie), nebst einer Aufgabensammlung. 3te Auflage. Leipzig, Teubner, 1907. 8vo. 8 + 240 pp Cloth. M. 3.30

MAIR (D.). School course in mathematics. London, Frowde, 1907. 12mo. 388 pp. Cloth. 3s. 6d.

MILNE (W. J.). High school algebra, embracing a complete course for high schools and academies. New York, American Book Company, 1907. 12mo. 391 + 34 pp Cloth. $1.00

NAPRAVNIK (F.). Vollständig gelöste Maturitatsaufgaben aus der Mathematik. Fur Schüler der obersten Klassen an Realschulen und Gymnasien sowie zum Selbststudium. Wien, Deuticke, 1907. 8vo. 3 + 233 pp. M. 3.00

OLNEY (E.). The complete algebra for high schools, preparatory schools and academies. New edition. New York, American Book Company, 1907. 12mo. 12 + 439 pp. (Olney's new series.) Cloth $1.10

ORTEGA Y SALA (D. M.). Geometria. Obra elegida de texto para el ingreso en las Academias Militares, en el concurso celebrado en 30 de Abril del mismo año por la Dirección General de Instrucción Militar, para el ingreso en la Escuela Naval Flotante, etc. Vol. II. Ejercicios resueltos. Madrid, Perlado, 1907. 8vo. 374 pp. P. 15.00

SCHULTZE (A.). Advanced algebra with answers. New York, Macmillan, 1906. 12mo. 7 + 562 + 76 pp. Half leather. $1.25

——. Elementary algebra. New York, Macmillan, 1906. 12mo. 11 + 396 + 63 pp. Half leather. $1.10

STAINER (W. J.). Junior practical mathematics. Part I, with answers. London, Bell, 1907. 12mo. 1s. 6d.

——. Part II, with answers. London, Bell, 1907. 12mo. 2s. 6d.

TREUTLEIN (P.). See HENRICI (J.).

III. APPLIED MATHEMATICS.

AUBRY (C.). Les nouvelles méthodes de calcul de la poussée des terres, basées sur la théorie de l'élasticité et ses applications. Paris, Germain, 1906. 8vo. 143 pp.

BOUSSINESQ (J.). Propagation du mouvement autour d'un centre, dans un milieu élastique homogène et isotrope. Paris, Gauthier-Villars, 1906. 4to. 38 pp. F. 2.00

GAUSS (F. G.). Trigonometrische und polygonometrische Rechnungen in der Feldmesskunst. 3te Auflage. Hefte 6–9. Halle, Strien, 1906. M. 10.50

HELMHOLTZ (H. von). Vorlesungen uber theoretische Physik. Herausgegeben von A. König, O. Krigar-Menzel, M. Lane, F. Richarz, C. Runge. Band IV: Vorlesungen über Elektrodynamik und Theorie des Magnetismus. Herausgegeben von O. Krigar-Menzel und M. Lane. Leipzig. Barth, 1907. 8vo. 10 + 406 pp. M. 16.00

HOLZMÜLLER (G.). Elementare kosmische Betrachtungen über das Sonnensystem und Widerlegung der von Kant und Laplace aufgestellten Hypothesen über dessen Entwicklungsgeschichte. Einige Vorträge. Leipzig, 1906. 6 + 98 pp. M. 1.80

HUTCHINSON (R. W.). Long-distance electric power transmission; being a treatise on the hydro-electric generation of energy, its transformation, transmission and distribution. New York, Van Nostrand, 1907. 12mo. 345 pp. Cloth. $3.00

JOYCE (S.). Examples in electrical engineering. New and improved edition. London, Longmans, 1907. 12mo. 248 pp. Cloth. 5s.

KOENIGSBERGER (L.). Hermann von Helmholtz: translated by Frances A. Welby with a preface by Lord Kelvin. Oxford, University Press (American branch), 1906. 8vo. 17 + 440 pp. Cloth. $5.25

LOVE (A. E. H.). Lehrbuch der Elastizität. Autorisierte deutsche Ausgabe, unter Mitwirkung des Verfassers besorgt von A. Timpe. Leipzig, 1907. 8vo. 16 + 664 pp. Cloth. (B. G. Teubner's Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Vol. 24.) M. 16.00

POINCARÉ (H.). Leçons de mécanique céleste professées à la Sorbonne. Vol. 2. 1re partie: Développement de la fonction perturbatrice. Paris, Gauthier-Villars, 1907. 8vo. 172 pp. F. 6.00

SAUSSURE (R. de). Théorie géométrique du mouvement des corps. Fin de la 1re partie et commencement de la 2me partie. La géométrie des feuillets. Genève, Kundig, 1906. 109 pp.

VOGEL (E.). Ueber die mechanische Ermittlung des Durchdringungspolygons. (Progr.) Tetschen, 1906. 8vo. 3 pp.

WITTENBAUER (F.). Aufgaben aus der technischen Mechanik. Band I: Allgemeiner Teil. 770 Aufgaben nebst Lösungen. Berlin. Springer, 1907. 8vo. 10 + 289 pp. M. 5.00

WRIGHT (T. W.). Elements of mechanics, including kinematics, kinetics, and statics, with applications. 7th edition, revised. New York, Van Nostrand, 1907. 8vo. 382 pp. Cloth. $2.50

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THE FEBRUARY MEETING OF THE SAN FRANCISCO SECTION.

THE eleventh regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY was held at Stanford University on February 23, 1907. The first session opened at eleven o'clock, with Professor E. J. Wilczynski, of the University of California, in the chair. The attendance was eighteen, including the following twelve members of the Society:

Professor R. E. Allardice, Professor H. F. Blichfeldt, Professor R. L. Green, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Dr. W. A. Manning, Professor H. C. Moreno, Professor C. A. Noble, Dr. T. M. Putnam, Professor Irving Stringham, Professor E. J. Wilczynski.

The next meeting of the Section will be held at the University of California on September 28, 1907.

The following papers were read at the February meeting:

(1) Professor E. J. WILCZYNSKI: "Projective differential geometry of curved surfaces."

(2) Professor H. F. BLICHFELDT: "Concerning some properties of surfaces of revolution."

(3) Professor G. A. MILLER: "On the groups generated by two operators of order three whose product is of order four."

(4) Professor L. M. HOSKINS: "The practical formulation of the laws of motion."

(5) Professor R. E. ALLARDICE: "On a limit of the roots of an equation that is independent of all but two of the coefficients."

(6) Mr. E. T. BELL: "Note on linear congruences."

(7) Dr. W. A. MANNING: "On the limit of the degree of primitive groups."

Mr. Bell was introduced by Professor Blichfeldt. In the absence of Professor Miller, his paper was read by title. Abstracts of the papers follow below.

1. Professor Wilczynski's paper introduces a number of new conceptions into the theory of surfaces and traces their interrelations. One of the most interesting of these is as follows. Through every point P of a surface S there pass two asymptotic curves A_1 and A_2. Construct the locus of the asymptotic tan-

gents of S belonging to the points of A_1. We thus obtain, besides the developable D_1 of which A_1 is the cuspidal edge, a ruled surface R_1 upon which A_1 is an asymptotic line. Similarly for A_2. The osculating hyperboloids of R_1 and of R_2 at P coincide. The osculating linear complexes of the four surfaces P_1, R_1, D_2, R_2 are so related that each is in involution with two of the others. Consider in particular the osculating linear complexes of D_1 and D_2. The directrices of the linear congruence common to them are of special interest. One of them lies in the plane tangent to S at P, while the other passes through P. Since there are two such directrices associated with every point P of S, there are thus obtained, associated with the surface S, two congruences. Consider for instance the congruence of the directrices of the first kind, those which pass through P. There are two directions through P such that the directrices of the same kind belonging to points of S infinitesimally close to P in those directions will intersect the directrix of P. In this way two families of ∞^1 curves are defined on S, two passing through every point. These are called the directrix curves. The directrices of the second kind, treated from a similar point of view, yield the same curves on S.

The non-homogeneous equation of a non-ruled surface may be expressed by a development of the form

$$z = xy + \tfrac{1}{6}(x^3 + y^3) + \tfrac{1}{24}(Ix^4 + Jy^4) + \cdots,$$

where I and J are absolute projective invariants. The complete geometric interpretation of this canonical form is given and is found to depend upon the properties of certain cubic surfaces which have contact of the third order with S at P, and which are completely determined by certain further geometric properties.

Analytically the treatment is based upon the theory of the invariants of a system of two partial linear differential equations of the second order with two independent variables.

2. Representing the plane conformally upon any surface, the straight lines of the plane will give rise to a system of curves on the surface, the study of which is interesting for the reason that they may be taken to represent straight lines in a plane geometry in which the euclidean axioms hold, with the exception of the "superposition" axiom. In particular, as Professor Blichfeldt points out, if the surface taken be a certain surface of revolution, some interesting results follow.

3. Professor Miller's paper is devoted to the study of some of the groups which are generated by two operators s_1, s_2 of order three whose product is of order four. The main results are as follows: The order of the commutator of s_1, s_2 must always be greater than 2. If the order of $s_1s_2^2$ is 3, the group $\{s_1, s_2\}$ generated by s_1, s_2 is the group of order 48 which involves operators of order 4 but no subgroup of order 24. If two operators satisfy the conditions $s_1^3 = s_2^3 = (s_1s_2)^4 = (s_1s_2^2) = 1$, they generate the simple group of order 168 unless each of them is identity. If two operators whose commutator is of order 5 satisfy the conditions $s_1^3 = s_2^3 = (s_1s_2)^4 = (s_1s_2^2)^5 = 1$, they generate the simple group of order 360. When two operators of order 2 have a commutator of even order, they generate a group whose order is four times the order of their commutator; when they have a commutator of odd order, they generate a group whose order is either twice or four times the order of their commutator. If an operator of order 2 and an operator of order 3 have a commutator of order 2, they generate either the tetrahedral group or the direct product of this group and the group of order 2. If two operators of order 3 have a commutator of order 2, they generate a group whose order is one of the four numbers 12, 36, 144, 288. The eight commutators which may be obtained from a single commutator by permuting its elements have a common order.

4. Professor Hoskins's paper is in abstract as follows: By a "practical" formulation of the laws of motion is meant a form of statement suited to the point of view of the average intelligent student pursuing a first course in mechanics. Such a formulation must not be palpably unsound, but neither can it aim at strict logical rigor. It must be based upon preliminary notions derived from ordinary experience, and must not raise highly abstract questions or logical subtleties which can be appreciated only by the exceptional mind and after mature study and reflection. The points especially emphasized in the paper are (1) that force should be taken as a fundamental quantity (contrary to the practice of some writers who aim at logical rigor); (2) that the simple fact that every force acting upon a body is exerted by some other body should not be left to inference but should be an explicit part of the definition of force, and should be further emphasized in the statement of Newton's first and third laws; (3) that the law of action and

reaction should be restricted to forces, and should state explicitly that two bodies and only two are concerned in every "action and reaction"; and (4) that the law of composition of forces should be accepted as a fundamental principle, being neither implicitly contained in Newton's second law nor otherwise susceptible of deductive proof. This fourth point is urged both on the ground of intelligibility to the beginner and on that of scientific soundness. The paper does not attempt a full discussion of the question as to the best method of presenting an elementary course in mechanics. The formal statement of the laws of motion is not the best starting point for such a course, but on the contrary should come somewhat late. The practical value of such a formal statement is to sum up in convenient form what is already more or less familiar to the student.

5. Professor Allardice gave a proof of the following theorem: Every equation of the form $ax^n + bx^m + cx^l + \cdots + a_1x + a_0 = 0$ has a root whose modulus is not greater than

$$\left|\frac{a_0}{a_1}\right| \cdot \frac{n}{n-1} \cdot \frac{m}{m-1} \cdot \frac{l}{l-1} \cdots$$

whatever be the values of the coefficients $a, b, c, \cdots$.

6. Instead of proceeding as usual in the solution of a linear congruence in n indeterminates, Mr. Bell makes use of the theory of partitions. The absolute term c has added to it multiples of the modulus, and the resulting absolute terms $c_0, c_1, \cdots, c_k$ (where $c_i = c + im + i$ ranges from 0 to the integral part of the fraction $(nm - c)/m$, where m is the modulus) are partitioned in all possible ways into m parts, and from the solution of m congruences in one indeterminate the (incongruent) solutions of the original congruence are obtained. Further, the solutions fall into "classes," according to the value of i, and each "class" gives rise to a number of "sets" of solutions, the number of sets being the number of partitions of $c + im$ into n parts, no part being greater than m. The number of solutions of the original congruence in each "set" is easily determined and hence partition formulas for the number of solutions (incongruent, mod m) are obtained. These may be utilized to deduce theorems in partitions by equating them to the known formulas as given, *e. g.*, by Smith ("Congruences and Equations," Col-

lected Works, volume 1). Also the known criteria for the resolubility of such congruences can be put in partition forms.

7. The limit of the degree of a simply transitive primitive group which does not include the alternating group of that degree and which contains a given substitution is the object of investigation in Dr. Manning's paper. If the given substitution is of the (odd) prime order p and of degree pq, such a limit is given by the formula

$$(p+q)\left\{q+\frac{p-1}{p+1}+\mu(q-1)\right\}-\frac{p+\theta q}{\theta-1}\theta^{\mu},$$

where $\theta = 2p/(p-1)$ and μ is the characteristic of $\log_\theta q$.

W. A. Manning,
Secretary of the Section.

THE FIFTY-SEVENTH MEETING OF THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE.

The fifty-seventh meeting of the American Association for the Advancement of Science was held in New York City during the convocation week December 27, 1906, to January 2, 1907. The president of the meeting was Dr. W. H. Welch, of Johns Hopkins University. The address of the retiring president, Professor C. M. Woodward, entitled "The Science of Education," was given in the auditorium of the Horace Mann School of Columbia University on the evening of the opening day. This address has been published in full in *Science* for December 28, 1906.

The meetings of Section A were not well attended, only about twenty members being present at the organization meeting. The officers of the section were: vice-president, Edward Kasner; secretary, L. G. Weld; councilor, G. B. Halsted; member of the general committee, James McMahon; press secretary, the secretary of the section; sectional committee, Ormond Stone, E. B. Frost, E. O. Lovett, Harris Hancock, A. N. Skinner, together with the vice-president and the secretary of the section.

The following mathematicians and astronomers were, upon nomination by the sectional committee, elected by the council to fellowship in the Association:

R. H. Baker, G. L. Brown, R. S. Dugan, J. B. Faught, Fannie C. Gates, O. E. Glenn, W. J. Graham, W. A. Granville, S. M. Hadley, Henrietta L. Leavitt, Percival Lowell, James Maclay, H. R. Manning, G. D. Olds, G. A. Plimpton, C. L. Poor, N. C. Riggs, L. G. Schultz, F. H. Smith, A. C. Washburne, N. R. Wilson, Anna S. Young.

The next annual meeting of the Association will be convened in Chicago on Thursday, December 26, 1907, under the presidency of Professor E. L. Nichols, of Cornell University. Professor E. O. Lovett will be vice-president of Section A.

The address of the retiring vice-president, Dr. W. S. Eichelberger, entitled, "Clocks, ancient and modern," was presented on the afternoon of Thursday, December 27, in Fayerweather Hall of Columbia University. This address has already been published in *Science* for March 22 of the present year.

A joint session of Section A with the AMERICAN MATHEMATICAL SOCIETY and the Astronomical and Astrophysical Society of America was held on Friday forenoon, December 28, in Schermerhorn Hall. The chair was occupied by Professor Simon Newcomb, past president of each of the participating societies. This was perhaps the most largely attended and the most generally interesting of any of the meetings in which any of the participating societies had a part. The following programme was presented: numbers (1) and (5) being contributed by the Mathematical Society; (2) and (4) by the Astronomical and Astrophysical Society; (3), (6) and (7) by Section A.

(1) Professor S. E. SLOCUM: "The rational basis of mathematical pedagogy."

(2) Professor E. E. BARNARD: "Photographic observations of the milky way."

(3) Professor JAMES MCMAHON: "The stream function for a straight channel with a circular island."

(4) Professor W. H. PICKERING: "The tenth satellite of Saturn."

(5) Dr. F. L. GRIFFIN: "On the law of gravitation in the binary systems."

(6) Professor C. L. DOOLITTLE: "Latitude terms of long period."

(7) Professor EDWARD KASNER: "Dynamical trajectories."

Abstracts of (1) and (5) of the above papers appear in the current volume of the BULLETIN, pages 265, 266; abstracts of (2), (4) and (6) appeared in *Science*, April 12 and 19, in the report of the eighth annual meeting of the Astronomical and Astrophysical Society of America. The other papers are further noticed below.

The full list of papers appearing upon the programme of Section A, with abstracts of such as deal with purely mathematical subjects, is given below.

(1) Professor C. L. DOOLITTLE: "An examination of the results of seven years' observation with the zenith telescope of the Flower Observatory for latitude terms of long period."

(2) Mr. PHILIP FOX: "A preliminary report on a solar rotative period investigation."

(3) Mr. A. O. GRANGER: "The retrograde motion of Phoebe."

(4) Professor G. B. HALSTED: "The sect-carrier and the set-sect."

(5) Professor HARRIS HANCOCK: "On a fundamental theorem of Weierstrass by means of which the theory of elliptic functions may be established."

(6) Professor EDWARD KASNER: "Dynamical trajectories."

(7) Professor JAMES MCMAHON: "The stream function for a straight channel with a circular island."

(8) Dr. S. A. MITCHELL: "Preliminary wave-lengths of flash spectra taken in Spain, August 30, 1905."

(9) Professor G. A. MILLER: "On the minimum number of operators whose orders exceed two in any finite group."

(10) Professor DAVID TODD: "Results of physical observations on the saturnian system with the 18-inch Clark refractor of the Amherst College observatory."

4. The school of Plato fixed as the instruments for the solution of geometric problems, the ruler and compasses, the straight line and circle. As in euclidean geometry the straight line is a circle (of infinite radius) and as Euclid unconsciously made in his very first proposition the 'assumption of the compasses,' that "If a circle have a point within and a point without another circle, it has two points on this other" (Halsted, Rational Geometry, VI, 2), the world has had to await the coming of the non-euclidean geometry to become conscious of the fact that elementary geometry has been carrying a wholly unnecessary 'rider.'

The compasses may be superseded by the simpler "transferrer of line-segments," for which the name "sect-carrier" has been adopted. Thus without the circle or compasses all the problems of elementary geometry are solved in the first edition of Halsted's Rational Geometry. But a remarkable additional simplification has now been achieved, and this paper makes public for the first time the simple demonstration which makes it available for the elements of geometry. This advance is the substitution of the set-sect for the sect-carrier. The transference of only a single sect need be assumed for the solution of all the problems of elementary geometry. Consequently the power to take a centimeter on a given straight line is found to be assumption enough to supersede the circle, the compasses, and even the sect-carrier. Nothing now is needed but a ruler and a set-sect.

5. The theorem discussed by Professor Hancock is stated by Weierstrass in the "Theorie der abelschen Funktionen" (*Crelle's Journal*, volume 52, § 7; and Mathematische Werke, volume 1, page 349).

By means of this theorem it may be shown that the $\wp$-function may be expressed as the quotient of two series which are both convergent for all values of the variable; the same is true of the functions $\sqrt{\wp u - e_\lambda}$ $(\lambda = 1, 2, 3)$. It follows directly from Weierstrass's theorem that the σ-function may be expressed as a convergent series for all values of the variable.

The different series are calculated and it is interesting to compare the results usually obtained from the well known theorem, also due to Weierstrass, that every one-valued function that has not an essential singularity in the finite portion of the plane may be expressed as the quotient of two power series which are convergent for all values of the variable.

Weierstrass's theorem is also generalized and applied to differential equations of a higher order.

6. Professor Kasner discussed two general questions of interest in connection with celestial mechanics relating to the geometry of dynamical trajectories. The first is suggested by the problem of binary stars and Bertrand's discussion of the interdependence of Kepler's laws. It is shown that two distinct fields of force can have only a certain multiplicity of trajectories in common. It is then possible to determine a field from

a minimum number of trajectories. In particular the newtonian law may be deduced without assuming, as Bertrand does, that all the orbits are conics.

The second part of the paper relates to the problem of n bodies and extends some of the results which hold for a single particle (see *Transactions of the American Mathematical Society*, 1906, 1907). For example, the locus of the centers of the osculating spheres, under prescribed initial conditions, is a cubic curve; in the case of a single particle, on the other hand, it is a straight line. The results obtained are true for all interacting particles.

7. Professor McMahon's paper dealt with one of the standing problems in two-dimensional fluid motion. A solution is here obtained by imagining a doublet placed midstream in a uniform current so that the line from the source to the adjoining sink points in the direction of the undisturbed current. The appropriate stream function is determined to suit the boundary conditions by the image method, and it is shown that one of the stream lines breaks up into the median line of the channel and a symmetric oval. The strength of the doublet can be so adjusted that this oval does not differ appreciably from a given circle when the latter does not occupy more than half the breadth of the channel.

9. Professor Miller's paper appeared in full in the February BULLETIN, pages 235–239.

With the exception of those papers which appear upon the joint programme of Friday morning, as given above, the papers of Section A were presented in connection with the programme either of the Mathematical or of the Astronomical Society, according to the subject matter treated in each case. This arrangement was made in accordance with a resolution adopted at the Ithaca meeting to the effect that "the sectional committee be empowered to turn over technical papers to the technical societies."

LAENAS GIFFORD WELD,
Secretary.

THE STATE UNIVERSITY OF IOWA,
IOWA CITY, IOWA.

ON A FINAL FORM OF THE THEOREM OF UNIFORM CONTINUITY.

BY PROFESSOR E. R. HEDRICK.

(Read before the American Mathematical Society, February 23, 1907.)

1. It is one of the fundamental theorems of analysis that a function which is continuous in a closed interval is uniformly continuous in that interval. A natural generalization to any closed assemblage is plausible and has been proved by Jordan;[*] again, an extension to a function of any number of variables is almost obvious. Another and a less obvious generalization has occurred to the writer, of which a special case has been stated by Baire.[†] In the case of a function of one variable this may be stated as follows:

Let $f(x)$ *be a function defined on any assemblage* (H) *of values of* x, *and let* (E) *be a closed subassemblage of* (H');[‡] *if the oscillation* $\Omega(x)$ *is defined with respect to the values of* $f(x)$ *on* (H) *in the usual manner*[§] *and if* $\Omega(x) \leqq k$ *at each point of* (E), *then, corresponding to any positive number* ϵ, *there exists another positive number* η, *such that* $|f(x)-f(\xi)| < k+\epsilon$ *when* ξ *is any point of* (E) *and* x *is any point of* (H) *for which* $|x-\xi| < \eta$.

The nature of the extension will appear directly; at this point attention is directed to the final character of this result, in its application to any function for values assumed on any assemblage.

2. The ordinary statements may be revised by use of the concept of oscillation, and it is desirable for what follows to do this. If a function $f(x)$ is defined on any assemblage (H) whose limiting points form another assemblage (H'), the values

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[*] Jordan, Cours d'analyse, 2d ed., vol. 1, p. 48.

[†] Baire, Thesis: "Sur les fonctions de variables réelles," *Annali di Matem.*, 1899, p. 15; Baire, Leçons sur les fonctions discontinues, Paris, 1905; Borel, Leçons sur les fonctions de variables réelles, Paris, 1905, p. 27. Baire's statement makes (H) a continuum and $(E)=(H)$. See also W. H. Young, Theory of sets of points, Cambridge, 1906, p. 218.

[‡] (H') is the assemblage of all the limiting points of (H), *i. e.*, the *first derived assemblage*. It is not necessary that (E) be part of (H'), but the real content of the theorem is the same if this restriction is made.

[§] See § 2.

of $f(x)$ at points of (H) near a point h' of (H'), for example in the interval $(h'-\delta,\ h'+\delta)$, have lower and upper limits which may be denoted by

$$l[h',\ (H),\ \delta, f(x)] \quad \text{and} \quad L[h',\ (H),\ \delta, f(x)],$$

respectively, at least provided $f(x)$ is limited on (H).* Let $\omega[h',\ (H),\ \delta, f(x)] = L - l$ be called the oscillation of $f(x)$ on (H) in the interval $(h'-\delta, h'+\delta)$. When there is no ambiguity we shall write simply $L(h', \delta)$, $l(h', \delta)$, $\omega(h', \delta)$, respectively. Again, $\omega(h', \delta)$ is never negative, and it does not increase with δ; hence it has a lower limit $\Omega(h')$, which is defined at every point of (H'). $\Omega(h')$ is also the difference between the lower limit of $L(h', \delta)$ and the upper limit of $l(h', \delta)$, which we shall denote by $M(h')$ and $m(h')$ respectively.

It is evident that if (H) is an interval $a \leqq x \leqq b$, $f(x)$ is continuous at every point at which $\Omega(x) = 0$ and discontinuous at every point at which $\Omega(x) \neq 0$.

Hence the ordinary theorem may be stated as follows: *If* (H) *is an interval* $a \leqq x \leqq b$ *and if* $\Omega(x) = 0$ *for every* x, *then corresponding to every positive number* ϵ *there exists another positive number* η *independent of* x, *such that* $\omega(x, \delta) < \epsilon$ *whenever* $\delta < \eta$.

Jordan's form requires $\Omega(x) = 0$ for every x in a certain closed assemblage (E) of the interval, and shows that $\omega(x, \delta) < \epsilon$ for $\delta < \eta$ if x belongs to (E).

3. The proof of the theorem stated in § 1 is now easy. For if $\Omega(h') \leqq k$ for all points h' in (H') which belong to a closed subassemblage (E) of (H'), then corresponding to the given ϵ and for a fixed point e of (E) we can surely select $\eta(e)$ dependent on e and such that

$$\omega[e,\ (H),\ \delta, f(x)] < k + \epsilon \quad \text{whenever } 0 < \delta \leqq \eta(e),$$

since Ω is the lower limit of ω. Suppose now that $\eta(e)$ has a lower limit zero. Then there surely exists a monotone sequence of e's : $e_1,\ e_2,\ \cdots,\ e_n,\ \cdots$ such that

$$\lim_{i=\infty} \eta(e_i) = 0.$$

Let $\bar{e}$ be the single limiting point of this sequence; it lies in (E) since (E) is closed. Hence

$$\omega[\bar{e},\ (H),\ \delta, f(x)] < k + \epsilon \quad \text{whenever } 0 < \delta \leqq \bar{\eta},$$

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* Even this provision is unnecessary if one desires to introduce the ideal values ± ∞.

where $\bar{\eta} = \eta(\bar{e})$. Let e_m be any e_i within the interval $(\bar{e} - \bar{\eta}/2, \bar{e} + \bar{\eta}/2)$; then surely we shall have

$$\omega\left[e_m, (H), \frac{\delta}{2}, f(x)\right] \leqq \omega[\bar{e}, (H), \delta, f(x)] < k + \epsilon$$
$$\text{whenever } \delta < \bar{\eta},$$

which is in contradiction with the assumption that $\eta(e_m)$ necessarily approaches zero as m becomes infinite.*

Let η denote the lower limit of $\eta(e)$ for points e in (E). Since $\eta \neq 0$ and since $\eta(e) \geqq \eta$ for every e in (E) it follows that

$$\omega[e, (H), \delta, f(x)] < k + \epsilon \quad \text{whenever } 0 < \delta < \eta,$$

where η is independent of e; and the theorem is proved: *If* $f(x)$ *is defined on* (H), *and if* $\Omega(x) \leqq k$ *for every* x *in a closed subassemblage* (E) *of* (H'), *then* $\omega[e, (H), \delta, f(x)] < k + \epsilon$ *whenever* $0 < \delta < \eta$, where e *is any point of* (E), *the order of choice being* ϵ, η, e.

This restatement is somewhat clearer from the standpoint of the oscillation.† It is evident that the theorem and the proof may be stated for a function of any number of variables without essential modification of either the notations or the arguments.

COLUMBIA, MO.,
February, 1907.

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* This conclusion may also be reached, but hardly more simply, by applying an extension of Borel's theorem (Borel, l. c., p. 9) on the possibility of covering an interval by a finite number of intervals, if we first extend Borel's theorem to fit the case of intervals enclosing the points of any closed assemblage, as is obviously possible.—(Note, March 15.) My attention is called by Professor J. W. Young to the fact that this extension of Borel's theorem is stated specifically by Veblen, BULLETIN, vol. 10, p. 436; and by W. H. Young, Theory of sets of points, Cambridge, 1906, p. 41.

† The oscillation in an interval, $\omega(x, \delta)$ approaches its limits $\Omega(x)$ (the oscillation at a point), *uniformly* if $\Omega(x) = 0$. A mistaken idea may suggest itself that ω approaches its limit Ω uniformly in any case; this is not true, as is seen in the case of functions which are continuous at least once in every interval and also discontinuous at least once in every interval; the theorem stated is all the more remarkable on this account.

THE GROUPS GENERATED BY THREE OPERATORS EACH OF WHICH IS THE PRODUCT OF THE OTHER TWO.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, February 23, 1907.)

LET s_1, s_2, s_3 represent any three operators of a finite group G which satisfy the three conditions

$$s_1s_2 = s_3, \quad s_2s_3 = s_1, \quad s_3s_1 = s_2.$$

These give rise to the following equations:

$$s_1s_2=s_2^{-1}s_1=s_2s_1^{-1}=s_3,\ s_2s_3=s_3s_2^{-1}=s_3^{-1}s_2=s_1,\ s_3s_1=s_1^{-1}s_3=s_1s_3^{-1}=s_2.$$

From the first continued equation it follows that s_1 and s_2 transform each other into their inverses and have a common square. From the second and third similar results follow with respect to s_2, s_3 and s_1, s_3 respectively. Hence s_1, s_2, s_3 are three operators such that each is transformed into its inverse by the other two. As any set of operators which fulfill the condition that each one is transformed into its inverse by all the others generates either the hamiltonian group of order 2^a or the abelian group of this order* and of type $(1, 1, 1, \cdots)$, it follows that s_1, s_2, s_3 generate one of the following four groups: identity, the group of order 2, the four-group, or the quaternion group. That is, if s_1, s_2, s_3 satisfy the three conditions imposed on them at the beginning of this paragraph, G must be one of these four groups, and it is evident that these operators may be so chosen that G is any one of these four groups.

If the given conditions are replaced by:

$$s_1s_2 = s_3, \quad s_2s_3 = s_1, \quad s_1s_3 = s_2,$$

there results the following system of continued equations:

$$s_1s_2=s_2^{-1}s_1=s_1^{-1}s_2=s_3,\ s_2s_3=s_3s_2^{-1}=s_2s_3^{-1}=s_1,\ s_1s_3=s_1^{-1}s_3=s_1s_3^{-1}=s_2.$$

From the first one of these it follows that s_2 is transformed into its inverse by s_1 and that the two operators s_1, $s_1^{-1}s_2$ are of order 2 since each of them is equal to its inverse. From the second and third it follows that s_2 is also transformed into its inverse by s_3 and that s_3 is of order 2, as is also otherwise evident. Hence the group generated by s_1, s_2, s_3 in this case is dihedral and its order is twice the order of s_2. Moreover, it is evident that the order of s_2 is any arbitrary number, so that every possible dihedral group may be generated by three operators which satisfy the conditions given at the head of this paragraph.

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* *Quar. Jour. of Mathematics*, vol. 37 (1906), p. 287.

The two sets of conditions given in the preceding paragraphs differ only with respect to the order of the factors in the last equation. As the first one of these sets is transformed into itself by the cyclic group of degree three, it has only two conjugates under this symmetric group. The second set has six conjugates under this group, since it is transformed into itself only by identity. Hence the two sets of conditions which have been considered are equivalent to the eight possible similar ones obtained by permuting the three operators in every possible manner. This follows from the fact that the transform of any set of conditions has the same properties as the original set. The preceding results may therefore be stated as follows: *If three operators are such that the product of any two is equal to the third, they generate either a dihedral group or the quaternion group.* When the three equations obtained in this way do not admit the cyclic permutation of the operators, these operators generate a dihedral group, and every dihedral group can be generated by three such operators. When they admit this cyclic permutation, they generate one of the following four groups: identity, the group of order 2, the four-group, the quaternion group.

The simplicity of these results seems to make them useful in the study of other group properties. The present study of the given relations is due to the fact that Professor Royce asked me recently for an instance in which group operators, or some physical or geometric transformations, were such as to satisfy the conditions given at the head of this note together with an additional condition. It follows from the above that the only case where three operators of order 4 satisfy these conditions is furnished by the operators of the quaternion group.

UNIVERSITY OF ILLINOIS,
February, 1907.

A TABLE OF MULTIPLY PERFECT NUMBERS.

BY PROFESSOR R. D. CARMICHAEL.

(Read before the American Mathematical Society, February 23, 1907.)

A MULTIPLY perfect number is one which is an exact divisor of the sum of all its divisors, the quotient being the multiplicity.* The object of this paper is to exhibit a method for determining all such numbers up to 1,000,000,000 and to give a complete table of them. I include an additional table giving such other numbers as are known to me to be multiply perfect.

Let the number N, of multiplicity m ($m > 1$), be of the form

$$N = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n},$$

where $p_1, p_2, \cdots, p_n$ are different primes. Then by definition and by the formula for the sum of the factors of a number, we have

$$(1)\quad m = \frac{p_1^{a_1} + p_1^{a_1-1} + \cdots + p_1 + 1}{p_1^{a_1}} \cdots \frac{p_n^{a_n} + p_n^{a_n-1} + \cdots + p_n + 1}{p_n^{a_n}}.$$

Hence

$$(2)\qquad m < \frac{p_1}{p_1 - 1} \cdot \frac{p_2}{p_2 - 1} \cdots \frac{p_n}{p_n - 1}.$$

These formulas will be of frequent use throughout the paper.

Since $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \quad 17 \cdot 19 \cdot 23 = 223{,}092{,}870$, multiply perfect numbers less than 1,000,000,000 contain not more than nine different prime factors; such numbers, lacking the factor 2, contain not more than eight different primes; and, lacking the factors 2 and 3, they contain not more than seven different primes.

First consider the case in which N does not contain either 2 or 3 as a factor. By equation (2) we have

$$(3)\qquad m < \tfrac{5}{4} \cdot \tfrac{7}{6} \cdot \tfrac{11}{10} \cdot \tfrac{13}{12} \cdot \tfrac{17}{16} \cdot \tfrac{19}{18} \cdot \tfrac{23}{22} = \tfrac{676039}{331776}.$$

Hence $m = 2$; moreover seven primes are necessary to this value. Now, $5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 = 37{,}182{,}145$. Therefore, since we are not to consider numbers greater than 1,000,000,000, not more than one of the seven prime factors of N can occur to a power higher than the first. Then every numerator except one in the second member of equation (1) will contain the factor 2, while the denominators are all odd. Therefore, since m now equals 2, it is clearly impossible to satisfy equation (1).

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* The name "multiply perfect" was introduced by Lehmer, *Annals of Math.*, ser. 2, vol. 2, p. 103.

Next consider the case in which N contains the factor 3 but not the factor 2. By (2) we may show that $m = 2$ or 3. The value $m = 3$ is readily excluded by the method of the last paragraph. Let us examine the case $m = 2$. With the present values equation (1) becomes

$$(4) \qquad 2 = \frac{3^{a_1} + \cdots + 3 + 1}{3^{a_1}} \cdot \frac{p_2^{a_2} + \cdots + p_2 + 1}{p_2^{a_2}} \cdots \frac{p_n^{a_n} + \cdots + p_n + 1}{p_n^{a_n}}.$$

Since each denominator of (4) is odd the prime 2 can enter as a factor only once in the numerator of the second member. Therefore a_1 is even. If $a_1 > 8$ it is easy to show that N is greater than 1,000,000,000. The primes introduced when $a_1 = 8$ or $a_1 = 6$ in each case enable us to exclude these values with readiness. We have left then to consider only the values $a_1 = 4, 2$. Our work will be abridged as much as possible.

(1) If $a_1 = 4$, we have $p_2 = 11$. It is then easy to show that $a_2 = 2$ or 4. (*a*) If $a_2 = 2$, we have readily $p_3 = 7$, $p_4 = 19$, $a_3 \geqq 2$, $a_4 = 2$, $p_5 = 127$. (*b*) If $a_2 = 4$, we have $p_3 = 5$, $p_4 = 3221$, $a_4 = 1$, $p_5 = 179$, $a_5 \geqq 2$. Both cases exceed the limit.

(2) If $a_1 = 2$, we have $p_2 = 13$. The possible values of a_2 are 1, 2, 4. (*a*) If $a_2 = 1$, then $p_3 = 7$ and $a_3 = 2$ or 4. If $a_3 = 2$, we have $p_4 = 19$, $a_4 = 2$, $p_5 = 127$, $a_5 = 2$, $p_6 = 5419$. If $a_3 = 4$, $p_4 = 2801$ and $a_4 > 1$. Both cases exceed the limit. (*b*) If $a_2 = 2$, we have $p_3 = 61$ and $a_3 = 1, 2, 3$. The value $a_3 = 3$ is excluded by its bringing into the numerator of (4) too high a power of 2. If $a_3 = 2$, we have $p_4 = 97$, $a_4 = 1$, $p_5 = 7$. If $a_3 = 1$, we have $p_4 = 31$, $a_4 = 2$, $p_5 = 331$. Both cases again exceed the limit. (*c*) If $a_2 = 4$, $p_3 = 30941$, which at once makes N too great.

This completes the discussion. No perfect number has been found. Hence

There are no multiply perfect odd numbers $< 1,000,000,000$.[*]

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[*] There are no odd numbers known to be multiply perfect. See Encyklopädie der Mathematischen Wissenschaften, vol. I, (IC1), p. 578.

If the number N is even, it is easy to show from (2) that $m \not> 6$, and that if $m = 6$ nine primes are necessary. From the equation $2\ 3\cdot 5\cdot 7\ 11\ 13\cdot 17\ 19\cdot 23 = 223{,}092{,}870$ we readily see that when N contains 9 different prime factors every such factor except 2 and 3 enters to a power not higher than the first and that 2 enters to a power not higher than the third. It is easy to show that these conditions do not satisfy (1); therefore $m \not> 5$.

Now N cannot contain 2 to a power higher than the fifteenth without going beyond our assigned limit. By the method illustrated above for the case N odd and with the aid of the preceding deductions, I have taken each value of a_1 from 1 to 15 (a_1 being the exponent of 2 in the factors of N) and have thus determined all the multiply perfect numbers up to 1,000,000,000. To insure accuracy I have also computed these numbers by another method depending upon certain results which I have not yet published.

The first table below contains all the multiply perfect numbers up to 1,000,000,000. The second table contains certain such numbers above 1,000,000,000. These two tables embrace all the numbers known to the writer to be multiply perfect. According to Mersenne $2^{127}-1$ and $2^{257}-1$ are prime; but since this result is uncertain I do not include in the table $2^{126}(2^{127}-1)$ and $2^{256}(2^{257}-1)$ which are perfect if Mersenne is correct.

TABLE OF ALL MULTIPLY PERFECT NUMBER BELOW 1,000,000,000.

Number.	Multiplicity.	Number.	Multiplicity.
6	2	523776	3
28	2	2178540	4
120	3	23569921	4
496	2	33550336	2
672	3	45532800	4
8128	2	142990848	4
30240	4	919636480	3
32760	4		

TABLE OF CERTAIN MULTIPLY PERFECT NUMBERS ABOVE 1,000,000,000.

Number. .	Multiplicity.
$2^7\cdot 3^4\ 5\cdot 7\cdot 11^2\cdot 17\cdot 19$	5
$2^8\cdot 3\cdot 5\cdot 7\cdot 19\cdot 37\cdot 73$	4

$2^{10}\cdot 3^4\cdot 5\cdot 7\cdot 11^2\cdot 19\cdot 23\cdot 89$	5
$2^{13}\cdot 3\cdot 11\cdot 43\cdot 127$	3
$2^{14}\ 5\cdot 7\cdot 19\cdot 31\ 151$	3
$2^{14}\cdot 3\cdot 5\cdot 7\cdot 19\ 31\cdot 151$	4
$2^{15}\cdot 3^5\cdot 5^2 . 7^2\cdot 11\ 13\cdot 17\cdot 19\cdot 31\cdot 43\cdot 257$	6
$2^{16}\cdot 131071$	2
$2^{18}\cdot 524287$	2
$2^{30}\cdot 2147483647$	2
$2^{60}\cdot 2305843009213693951$	2

PRESBYTERIAN COLLEGE, ANNISTON, ALA.,
February, 1907.

THE SYMMETRIC GROUP ON EIGHT LETTERS AND THE SENARY FIRST HYPOABELIAN GROUP.

BY PROFESSOR L. E. DICKSON.

(Read before the Chicago Section of the American Mathematical Society, March 30, 1907.)

THE set of all transformations

$$(1)\qquad \xi_i'=\sum_{j=1}^{3}(\alpha_{ij}\xi_j+\gamma_{ij}\eta_j),\quad \eta_i'=\sum_{j=1}^{3}(\beta_{ij}\xi_j+\delta_{ij}\eta_j)\quad (i=1,2,3)$$

with integral coefficients taken modulo 2 which leave invariant the quadratic form

$$(2)\qquad \xi_1\eta_1+\xi_2\eta_2+\xi_3\eta_3$$

form a group G_0, called the (total) senary first hypoabelian group. It is a subgroup of the senary abelian linear group. The order of G_0 is (Linear Groups, page 206)

$$2(2^3-1)(2^4-1)2^4(2^2-1)2^2=8!.$$

The object of this note is to prove that G_0 is simply isomorphic with the symmetric group on eight letters.

We make use of the subgroup J_0 of G_0 obtained as the second compound of the general quaternary linear homogeneous group Q modulo 2 (Linear Groups, page 208). The process is analogous to the formation of the determinant of the sixth order, called the second compound of a determinant of the fourth order, by employing the 36 minors of order 2 of the latter. As shown by Jordan, Moore, and the writer, Q is simply isomorphic with the alternating group on eight letters (cf. Linear Groups, page 291). Forming the second compounds E'_i of the generators $E_1, \cdots, E_6$ (loc. cit., page 291) of Q, we get

$$E'_1 = \begin{matrix} 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{matrix}, \quad E'_2 = \begin{matrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{matrix}, \quad E'_3 = \begin{matrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{matrix},$$

$$E'_4 = \begin{matrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{matrix}, \quad E'_5 = \begin{matrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{matrix}, \quad E'_6 = \begin{matrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{matrix},$$

the variables being $\xi_1, \xi_2, \xi_3, \eta_3, \eta_2, \eta_1$, in order. By a general property of the second compound group, $E'_1, \cdots, E'_6$ leave (2) invariant. By their origin,

$$(3) \qquad \begin{matrix} E'_1 \sim (23)(12), & E'_2 \sim (34)(12), & E'_3 \sim (45)(12), \\ E'_4 \sim (56)(12), & E'_5 \sim (67)(12), & E'_6 \sim (78)(12). \end{matrix}$$

Let $(12) \sim S$, S having the notation (1). Since S shall be an abelian substitution of period 2, we have

$$\delta_{ij} = \alpha_{ji}, \quad \beta_{ij} = \beta_{ji}, \quad \gamma_{ij} = \gamma_{ji} \qquad (i, j = 1, 2, 3).$$

Then S is commutative with E'_4 if, and only if,

$$\alpha_{23} = \beta_{23} = \beta_{33} = 0, \quad \alpha_{12} = \beta_{12}, \quad \alpha_{13} = \beta_{13},$$

$$\gamma_{12} = \alpha_{21}, \quad \beta_{11} = \gamma_{11}, \quad \alpha_{33} = \alpha_{11} + \gamma_{11} + \alpha_{13}.$$

The hypoabelian condition $\sum \beta_{3i}\delta_{3i} = 0$ now gives $\alpha_{13} = 0$. Thus $\eta'_3 = \alpha_{33}\eta_3$, so that $\alpha_{33} = 1$, $\gamma_{11} = \alpha_{11} + 1$. The hypoabelian condition $\sum \alpha_{1i}\gamma_{1i} = 0$ gives $\alpha_{12}\alpha_{21} = 0$. The resulting substitution S is commutative with E'_5 and E'_2 if, and only if, it has the form

$$S_1 = \begin{matrix} \alpha_{11} & 0 & 0 & 1+\alpha_{11} & 0 & 1+\alpha_{11} \\ 0 & \alpha_{22} & 0 & 0 & \gamma_{22} & 0 \\ 1+\alpha_{11} & 0 & 1 & 1+\alpha_{11} & 0 & 1+\alpha_{11} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \beta_{22} & 0 & 0 & \alpha_{22} & 0 \\ 1+\alpha_{11} & 0 & 0 & 1+\alpha_{11} & 0 & \alpha_{11} \end{matrix}.$$

If $\alpha_{11} = 1$, S_1 affects only ξ_2 and η_2 and hence leaves (2) invariant only if it be identity or $(\xi_2\eta_2)$. The latter is not commutative with E'_3. Hence $\alpha_{11} = 0$. Then S_1 is commutative with E'_3 if, and only if, $\alpha_{22} = 1$, $\beta_{22} = 0$. The hypoabelian condition on the second line gives $\gamma_{22} = 0$. The resulting substitution is

$$(4) \quad \sum : \xi'_1 = \eta_3 + \eta_1, \quad \xi'_3 = \xi_1 + \xi_3 + \eta_3 + \eta_1, \quad \eta'_1 = \xi_1 + \eta_3.$$

Now $\sum$ is seen to be commutative with E'_6, while

$$(5) \quad E'_1\sum : \xi'_2 = \xi_2 + \xi_3 + \eta_3 + \eta_2, \quad \xi'_3 = \eta_3 + \eta_2, \quad \eta'_3 = \xi_3 + \eta_2,$$

is of period 2. But $\sum$ was shown to be of period 2 and to be commutative with $E'_2, \cdots, E'_5$. From these results and the fact that $E'_1, \cdots, E'_6$ satisfy Moore's generational relations* for the abstract form of the alternating group on eight letters, it follows formally that

$$\sum, \quad E'_1\sum, \quad E'_2\sum, \quad \cdots, \quad E'_6\sum$$

satisfy Moore's generational relations (loc. cit.) for the abstract form of the symmetric group on eight letters.

We can, of course, verify the last statement by direct computation and hence establish our theorem independently of the theorems employed above. We have

$$(6) \qquad E'_2\sum : \quad \xi'_2 = \eta_2, \quad \eta'_2 = \xi_2;$$

$$(7) \quad \begin{aligned} E'_3\sum : \quad & \xi'_2 = \xi_1 + \xi_2 + \xi_3 + \eta_3 + \eta_2, \quad \xi'_3 = \xi_1 + \eta_3 + \eta_2, \\ & \eta'_3 = \xi_1 + \xi_3 + \eta_2, \quad \eta'_1 = \xi_1 + \xi_3 + \eta_3 + \eta_2 + \eta_1; \end{aligned}$$

$$(8) \qquad E'_4\sum : \quad \xi'_1 = \eta_1, \quad \eta'_1 = \xi_1;$$

$$(9) \quad E'_5\sum : \quad \xi'_3 = \xi_1 + \eta_3, \quad \eta_3 = \xi_1 + \xi_3, \quad \eta'_1 = \xi_1 + \xi_3 + \eta_3 + \eta_1;$$

$$(10) \quad E'_6\sum : \quad \xi'_2 = \eta_3 + \eta_2, \quad \xi'_3 = \xi_2 + \xi_3 + \eta_3 + \eta_2, \quad \eta'_2 = \xi_2 + \eta_3.$$

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* *Proc. Lond. Math. Soc.*, vol. 28 (1896), p. 357. Cf. Linear Groups, p. 287.

Thus the transpositions (12), (23), $\cdots$, (78) correspond to (4), (5), $\cdots$, (10), respectively. In the standard notations for abelian substitutions the latter are, respectively,

$$M_1N_{31}Q_{31},\quad M_3N_{23}Q_{23},\quad M_2,\quad M_3Q_{31}N_{23}R_{13}Q_{23},\quad M_1,$$
$$M_3R_{31}Q_{31},\quad M_2N_{33}Q_{32}.$$

THE UNIVERSITY OF CHICAGO,
August, 1906.

DOUBLE POINTS OF UNICURSAL CURVES.

BY PROFESSOR J. EDMUND WRIGHT.

THE coordinates of a unicursal curve may be expressed as rational functions of a parameter. If we assume the curve to be of order n and use non-homogeneous coordinates, we have

$$x = a(\lambda)/c(\lambda), \quad y = b(\lambda)/c(\lambda),$$

where a, b, c are polynomials of order n in the parameter λ. For the double points two values of the parameter give the same values of x and y, and the usual method for their determination consists in finding pairs of values of λ and μ that satisfy the equations

$$a(\lambda)/c(\lambda) = a(\mu)/c(\mu), \quad b(\lambda)/c(\lambda) = b(\mu)/c(\mu).$$

After elimination of μ from these equations and division of the result by certain extraneous factors, an equation of order $(n-1)(n-2)$ in λ is obtained, and the roots of this equation combine in pairs to give the parameters of the $\frac{1}{2}(n-1)(n-2)$ double points. The process of solution however involves the solution of an equation of order $(n-1)(n-2)$.

Suppose now that a, b, c are polynomials in λ with real coefficients, *i. e.*, suppose the curve real, and write $\lambda + i\mu$ for λ. Let a be $A(\lambda, \mu^2) + i\mu A'(\lambda, \mu^2)$ and similarly for b and c. It is clear that $\lambda + i\mu$ gives for (x, y) the value

$$\left(\frac{A + i\mu A'}{C + i\mu C'}, \quad \frac{B + i\mu B'}{C + i\mu C'}\right)$$

and that $\lambda - i\mu$ gives

$$\left(\frac{A - i\mu A'}{C - i\mu C'}, \quad \frac{B - i\mu B'}{C - i\mu C'}\right).$$

These values are the same if

$$\frac{A'C - AC'}{C^2 + \mu^2 C'^2} = 0, \quad \frac{B'C - BC'}{C^2 + \mu^2 C'^2} = 0.$$

It is at once clear that the common values of λ, μ satisfying $A'C - AC' = 0$, $B'C - BC' = 0$ give the parameters $\lambda + i\mu$, $\lambda - i\mu$ of the double points, and in addition the values of λ, μ which make $C = 0$, $C' = 0$. Also, a real pair of values of λ, μ corresponds to a real isolated double point, whilst a real crunode is given by λ real and μ purely imaginary. In either case μ^2 is real and the two above equations are in λ and μ^2; hence each real pair of values of λ, μ^2 gives a real double point and each imaginary pair an imaginary double point. For a crunode μ^2 is negative, and for an isolated point it is positive.

The number of intersections of C and C' is $n(n-1)$, whilst the number of intersections of $A'C - AC'$ and $B'C - BC'$ is apparently $(2n-1)^2$. We shall now show that this number is in reality much less.

Suppose that

$$a(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots, \quad b(\lambda) = b_0\lambda^n + b_1\lambda^{n-1} + \cdots,$$
$$c(\lambda) = c_0\lambda^n + c_1\lambda^{n-1} + \cdots$$

and write

$$\lambda + i\mu = r(\cos\theta + i\sin\theta).$$

Then

$$\begin{aligned} A'C - AC' = \{ & [a_0 r^n \sin n\theta + a_1 r^{n-1} \sin(n-1)\theta + \cdots] \\ & \times [c_0 r^n \cos n\theta + c_1 r^{n-1} \cos(n-1)\theta + \cdots] \\ & - [c_0 r^n \sin n\theta + c_1 r^{n-1} \sin(n-1)\theta + \cdots] \\ & \times [a_0 r^n \cos n\theta + a_1 r^{n-1} \cos(n-1)\theta + \cdots]\} \div r\sin\theta \end{aligned}$$

$$\begin{aligned} = & (a_0c_1 - a_1c_0) r^{2n-2} + (a_0c_2 - a_2c_0) r^{2n-3} \frac{\sin 2\theta}{\sin\theta} \\ & + \left[(a_0c_3 - a_3c_0)\frac{\sin 3\theta}{\sin\theta} + (a_1c_2 - a_2c_1)\right] r^{2n-4} \\ & + \left[(a_0c_4 - a_4c_0)\frac{\sin 4\theta}{\sin\theta} + (a_1c_3 - a_3c_1)\frac{\sin 2\theta}{\sin\theta}\right] r^{2n-5} \\ & + \cdots, \text{ etc.,} \end{aligned}$$

$$= (a_0c_1 - a_1c_0)(\lambda^2 + \mu^2)^{n-1} + (a_0c_2 - a_2c_0)2\lambda(\lambda^2 + \mu^2)^{n-2}$$
$$+ [(a_0c_3 - a_3c_0)(3\lambda^3 - \mu^2) + (a_1c_2 - a_2c_1)(\lambda^2 + \mu^2)](\lambda^2 + \mu^2)^{n-3}$$
$$+ [(a_0c_4 - a_4c_0)(4\lambda^3 - 4\lambda\mu^2) + (a_1c_3 - a_3c_1)2\lambda(\lambda^2 + \mu^2)](\lambda^2 + \mu^2)^{n-3}$$

plus terms of order lower than $2n - 5$.

Hence $A'C - AC'$ is of order $2n - 2$ and has a multiple point of order $n - 1$ at each of the circular points at infinity. Similarly $B'C - BC'$ has multiple points at the circular points. The number of other intersections of these two curves is therefore

$$(2n - 2)^2 - 2(n - 1)^2 = 2(n - 1)^2.$$

Of these $n(n - 1)$ are accounted for, and there remain

$$2(n - 1)^2 - 2(n - 1) = (n - 1)(n - 2).$$

Now the equations contain only even powers of μ, and therefore if λ, μ be one intersection, λ, $-\mu$ is another. If μ^2 be eliminated from them, and the extraneous polynomial in λ arising from $C = 0$, $C' = 0$ be divided out, there remains an equation of order $\frac{1}{2}(n - 1)(n - 2)$ for λ. To each value of λ corresponds one double point. The corresponding value of μ^2 may in general be determined by elimination, and hence in general if λ be real the double point is real.

As an example we consider the unicursal cubic

$$x = \frac{a_0\lambda^3 + a_1\lambda^2}{c_2\lambda + c_3}, \quad y = \frac{b_1\lambda^2 + b_2\lambda}{c_2\lambda + c_3}.$$

$$A'C - AC' = a_0c_2(\lambda^2 + \mu^2)2\lambda + a_0c_3(3\lambda^2 - \mu^2)$$
$$+ a_1c_2(\lambda^2 + \mu^2) + a_1c_3 2\lambda.$$
$$B'C - BC' = b_1c_2(\lambda^2 + \mu^2) + b_1c_3 2\lambda + b_2c_3.$$

The equations for λ and μ give

(1) $$b_1c_2(\lambda^2 + \mu^2) + b_1c_3 2\lambda + b_2c_3 = 0,$$

(2) $$-4a_0b_1c_3(\lambda^2 + \mu^2) + a_1b_1c_2(\lambda^2 + \mu^2) + 2(a_1b_1c_3 - a_0b_2c_3)\lambda = 0.$$

Hence

$$\frac{2b_1c_3\lambda + b_2c_3}{b_1c_2} = \frac{2(a_1b_1c_3 - a_0b_2c_3)\lambda}{(a_1c_2 - 4a_0c_3)b_1}$$

is the equation for λ, and the value of λ from this equation, substituted in (1), gives the value of μ^2.

BRYN MAWR COLLEGE,
March, 1907.

THE MATHEMATICAL TABLETS OF NIPPUR.

Mathematical, Metrological and Chronological Tablets from the Temple Library of Nippur. By H. V. HILPRECHT. Vol. XX, part I, of Series A, Cuneiform texts, published by the Babylonian Expedition of the University of Pennsylvania, 1906. Pp. xviii + 70 + (30 + xv) plates. Price, $5.

UNTIL the middle of the last century the mathematics of the Babylonians was practically a mystery to modern European scholars. It was asserted, it is true, in a general way by many writers on the history of the science that our sexagesimal fractions originated in the valley of Mesopotamia. Formaleoni of Venice had suggested as early as 1788 that these were due both to the ancient division of the year into three hundred and sixty days and to the properties of the regular hexagon. Even the Greeks had assigned these "fractiones physicæ," as they were called in the middle ages, to the Babylonians, and before their use by such writers as Hypsicles and Ptolemy the story had long been current that Pythagoras had studied mathematics with the priests along the Euphrates, where the sexagesimal system was known. There were also numerous assertions in the later Middle Ages that even our Hindu-Arabic numerals were due to the Chaldeans, the Kaldâ of the monument inscriptions, who overran and subdued Babylonia in the first millenium, B. C. It was this tradition that led Tonstall to say, in 1522: "Qui a Chaldeis primum in finitimos, deinde in omnes pene gentes fluxit," and Recorde (c. 1542) to remark: "In that thinge all men do agree, that the Chaldays, whiche fyrste inuented thys arte, did set these figures as thei set all their letters. for they wryte backwarde as you tearme it, and so doo they reade."

All this was mere fiction, or but little more than tradition, and our knowledge of Babylonian mathematics may be said to have had but little scientific foundation until Rawlinson deciphered two small and imperfect cylinders found in 1854 by W. K. Loftus at Senkereh, the ancient Larsam (Larsa), on the Euphrates, and now preserved in the British Museum. Discovering the meaning of the key word IB–DI, square, Rawlinson was able to show that these cylinders contained tables of squares and cubes, written on the sexagesimal system. Professor Sayce put the date of the cylinders between 2300 and 1600 B. C., and thought that there had been a great library at Senkereh which would probably yield other mathematical material. Unfortunately this expectation has never been realized so far as Senkereh is concerned, although a few other similar cylinders have been found elsewhere. There has recently been uncovered at Nuffar (the ancient Nippur), however, a collection of unexpected richness, and particularly valuable for the light which the cylinders throw upon ancient mathematics.

The excavations at Nippur began in 1889, and for eighteen years Professor Hilprecht has been connected with the work. He has examined, with more or less care, over 50,000 cuneiform tablets thus far excavated there, and has secured for Constantinople and Philadelphia the best treasures of what may have been a great temple library which the Elamites twice destroyed, viz., about 2150 B. C., and about 1900 B. C. Some of these tablets are brick textbooks prepared by teachers and possibly deposited with other works in the temple library. Some have the teacher's model and the pupil's copy, and still others seem to show the erasure of the latter's work. At any rate, such are the dates and circumstances and interpretations which Professor Hilprecht assigns to his great discovery. There are not wanting, however, those who attack this view, some in a spirit of apparent fairness, and others with what appears to be a spirit of somewhat captious criticism. It is claimed that the tablets have no literary value, and that there is no evidence of a temple library, since some of the cylinders come from one part of the ancient city and some from another. One writer even weakens his cause by offering the remark that the temple may have had a system of Carnegie branches. It is also said that several important tablets did not come from Nippur at all, and an Arab's statement is taken against Professor Hilprecht's argument to prove this fact. While it would seem to a layman in matters Assyriological that there may have been two libraries at Nippur, as there were at Nineveh, one old and the other of later date, and that the statement of an Arab trader would not have much weight in a scientific controversy, it is certain that the bickering over seventeen tablets out of some fifty thousand should not obscure the fact that here is the greatest discovery of all time relating to Babylonian mathematics. It is probable that, with the support of such eminent scholars as Professor Zimmern of Leipzig, Professor Hommel of Munich, and Professor Winckler of Berlin, Professor Hilprecht will not feel that his labors have been in vain. With the whole controversy the student of mathematical history need have no concern whatever. Here are the cylinders; they are genuine; they are ancient; they reveal the science of Babylonia of the second or third millenium B. C., and whether there was a temple library, or whether an Arab told the truth, is from the mathematical standpoint a consideration of no moment.

It is the result of a study of over forty mathematical cylinders that appears in the volume under review. These cylinders include multiplication and division tables, tables of squares and square roots, a geometric progression, a few computations, and some work on mensuration. .

The multiplication tables are all arranged on the column plan, the same that was used by the medieval Italians in their "per colonna" operations. They also resemble the Italian tables in that the multiplicands are not successive, the European tables giving only those products needed for the current measures. Professor Hilprecht feels sure that this reason did not, however, influence the Babylonian mathematicians, although this hypothesis might easily be in harmony with his discovery that the multiplicands are all factors of 60^4. The tables are carried much farther than those found in medieval Europe, the multiplicands extending to 180,000.

More valuable than the multiplication tables are those of division, and of these the cylinders numbered 22 and 25 are the most interesting. In the former the quotients of 60^4 by 1, 2, ⋯, 18 are given, as follows:

IGI	1	GAL–BI	8,640,000	A–AN
IGI	2	GAL–BI	6,480,000	A–AN
IGI	3	GAL–BI	4,320,000	etc.

The meaning will be understood by knowing that IGI–GAL (literally, "having an eye," hence the decider, determinator, denominator) means, practically, the divisor or simply "divided by"; that BI means "its," and refers to the dividend, which in this case is the mysterious 60^4; and that A–AN means "each." Hence the second line may be translated, "60^4 divided by $2 = 6,480,000$ each." In other words we have here the equivalent of a table of unit fractions of 60^4, in which the first is not $\frac{1}{1}$, but $\frac{2}{3}$. Now why, asks Professor Hilprecht, was $\frac{2}{3}$ thus indicated as a multiplier? In reply, it seems not unreasonable to consider the correspondence between these Babylonian remains and the Egyptian mathematics of the (nearly) contemporary Ahmes. This at once throws a good deal of light upon both of the cylinders here discussed. This No. 22 simply means the first fraction of 60^4 is 8,640,000; the second, 6,480,000, and so on. Now this first fraction in the Egyptian hieroglyphic and hieratic systems seems, from Ahmes, to have been $\frac{2}{3}$, the only non-unit fraction known to him, and apparently the first one known to the Babylonians. It was the only one, save $\frac{1}{2}$, to have a special symbol in Egypt, and it probably had a special name, at least, in Babylonia. The very fraction name, IGI–GAL, "having an eye," suggests the eye-shaped fraction symbol of the Egyptians, and raises the question whether the original might not have been an eye instead of *ro*, a mouth. Nevertheless it must be admitted that the tablet in question does not contain the well-understood cuneiform fraction symbols for $\frac{1}{2}$, $\frac{2}{3}$, etc., and that there is, therefore, no direct connection between it and the Ahmes manuscript. As to the fraction symbol, too, the relation between the form of eye and mouth is probably merely fanciful, although the Semitic (as distinguished from the Sumerian) Babylonian and the Hebrew both use "mouth" for fraction, as the Egyptians did. Thus the Babylonian for $\frac{2}{3}$ is *shînipu*, probably from *šinâ* (two) and *pû* (mouth), the "two-fraction," a form also found in Hebrew.

The most interesting of all the cylinders, however, is No. 25, which is transliterated as follows:

Line 1:	125	720	Line 9:	2,000	18
2:	IGI–GAL–BI	103,680	10:	IGI–GAL–BI	6,480
3:	250	360	11:	4,000	9
4:	IGI–GAL–BI	51,840	12:	IGI–GAL–BI	3,240
5:	500	180	13:	8,000	18
6:	IGI-GAL–BI	25,920	14:	IGI–GAL–BI	1,620
7:	1,000	90	15:	16,000	9
8:	IGI–GAL–BI	12,960	16:	IGI–GAL–BI	810

Of this Professor Hilprecht has, with much ingenuity, unraveled part of the meaning, viz: that $60^4 \div 103,680 = 125$, which accounts for two figures in lines 1, 2, and similarly for other pairs. He also notices that $125 = 2 \cdot 60 + 5$, and that

3,600 ÷ 5 = 720, and similarly for other odd numbered lines. Now is there any further explanation? In reply it may be noticed that the unit fraction already mentioned, a form met not only among the Egyptians but among the early Greeks and other peoples about the Mediterranean, may have played a part hitherto unrecognized among the Babylonians. This is also seen in the table on page 27, where certain unit fractions of 195,955,500,000,000 are given, and in numerous other places. Therefore instead of lines 1, 2, meaning $60^4 : 103{,}680 = 125$, the interpretation may be as follows:

$$\frac{1}{60^2}\cdot\frac{1}{103{,}680}\cdot 60^4 = \frac{1}{60^2}\cdot 125 = \frac{1}{30} + \frac{1}{720}.$$

That is, lines 1, 2 give the important numbers in connection with all these unit fractions, viz.: 125, 720, and 103,680, assuming, of course, that the computer knew the nature of the problem with respect to the powers of 60. In the same way lines 3, 4 may mean

$$\frac{1}{60^2}\cdot\frac{1}{51{,}840}\cdot 60^4 = \frac{1}{60^2}\cdot 250 = \frac{1}{15} + \frac{1}{360},$$

and so on. In every case the denominator of the last unit fraction is given. *E. g.*, lines 7, 8 give the unit fractions $\frac{1}{15} + \frac{1}{15} + \frac{1}{15} + \frac{1}{90}$; lines 9, 10 give $\frac{1}{2} + \frac{1}{8}$, and so on. All of the unit fractions save the last are readily seen, at least after the initial $\frac{1}{30}$ is found, so that only the last denominator seemed necessary. It is evident, however, that the cylinder does not actually contain these unit fractions, and the explanation here suggested may be far from the true one; but it is consistent, and if not exact it seems to show at least a connection between the number concepts of Babylonia and the Mediterranean countries. Moreover the cylinder has three other features of interest: (1) Passing from lines 7, 8 to 9, 10 the fractions (if we consider them as such) would become, following an apparent law of doubling, $\frac{8}{15} + \frac{1}{45}$; but, to maintain the unit fraction idea, they are written $\frac{1}{2} + \frac{1}{18}$ the denominator 18 appearing in line 9, all of which recalls the Egyptian treatment; (2) The table gives both an increasing and decreasing geometric progression; (3) It is evidently based upon $60^4 = 12{,}960{,}000$ which, as Professor Hilprecht notes, underlies all the mathematical texts described in this work, and which is nothing less than the mystic Platonic number, the "lord of better and worse births," the number of days in the "magnus Platonicus annus" of 36,000 years (of 360 days each), and the number at the basis of all the multiplication and division tables of Nippur, Sippar, and the library of Ashurbânapal.

The historic interest in this Platonic number does not relate to any extent to the supposed mysticism involved in it, but to the fact that Plato not improbably received it from the Pythagoreans, and they from their master the tradition of whose sojourn by the Euphrates is thus in some slight degree confirmed. At any rate, the whole matter seems to show a relation between the East and the West, both in the underlying idea of the unit fraction and in the number mysticism of the ancient philosophers.

A further debt which historians of mathematics owe to Professor Hilprecht arises from his contribution to the subject of ancient geometry. The results of his investigation of several cylinders on mensuration, together, no doubt, with the Scheil-Eisenlohr investigation* of the Ine-Sin tablet show "that at this early period the Babylonians must have been familiar with the following theorems: 1. The area of a rectangle is equal to the product of its base and altitude. 2. The area of a square is equal to the square of its side. 3. The area of a right triangle is equal to one-half the product of its base and altitude. 4. The area of a trapezoid is equal to one-half the sum of its bases multipled by its altitude." He also shows that they must have known either how to find the volume of a parallelepiped (and hence a cube) or else of a circular cylinder (and hence the area of a circle).

Still another inference of great importance, and supported by evidence that seems sufficient, is that the Babylonians knew, in some form or other, the law of the expansion of $(a + b)^2$, although whether this was derived through a study of geometric forms or by induction with numbers, it is impossible as yet to tell.

A final point of much interest may be mentioned, viz., the fact that the subtractive principle of the Romans is also found in the Babylonian remains. The Romans spoke of nineteen as "undeviginti," writing it as IXX or XIX, and Professor Hilprecht gives no less than twelve cuneiform methods of indicating the same idea (20–1). It seems to be certain that the

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* Ein altbabylonischer Felderplan nach Mittheilungen von F. V. Scheil, herausg. u. bearb. von A. Eisenlohr, Leipzig, 1896.

Babylonians applied this principle to 2 as well as 1, and even to higher numbers.

The work is illustrated by numerous drawings of mathematical tablets, and by a series of carefully executed photographic plates.

Altogether there has not appeared since the publication of the Eisenlohr translation of Ahmes such a valuable contribution in the way of source material for the study of ancient mathematics. It is earnestly to be hoped that Professor Hilprecht will continue in this important line, and that he will be able to assist still further in clearing up a number of vexed questions relating to the early mathematics of Mesopotamia. In particular it would be helpful if he could throw some light upon ancient calculation,* upon the Babylonian abacus (if one existed), and upon the relation (if any) between the number names of Phoenicia, Egypt, and Babylon. It is also to be hoped that he may succeed in giving us some information about the mathematics of the Shumeri (?), those non-Semitic inhabitants of the Euphrates valley whose language, the Sumerian, should, in the natural course of events, have influenced the mathematical terminology of much of the ancient world. It is to these people that there seems due some of the first work in mathematics and astronomy, and it is probable that the sexagesimal system itself had birth among them.

DAVID EUGENE SMITH.

OSGOOD'S THEORY OF FUNCTIONS.

THE BULLETIN has received for review both parts of the first volume, now completed, of Professor W. F. Osgood's Lehrbuch der Funktionentheorie (Leipzig, Teubner, 1905 and 1907). Pending the publication of a critical review, which is in preparation, we present herewith the author's own lucid and interesting summary of the contents, translated, with Dr. Osgood's permission, from the preface.

In the first volume of this work it is our purpose to develop systematically the theory of functions upon the basis of the infinitesimal calculus, in intimate contact with geometry and with mathematical physics. The first special developments are found in the second section, Chapter 6, page 179; they concern principally functions that are singly valued in a given region of the plane and that possess a derivative. In connection with these are discussed the elementary functions for complex values of the argument and the linear transformations of a complex variable. As would be expected, conformal depiction is given a leading position at the very outset. After these preliminaries the Cauchy integral theorem is introduced, and with it the whole cycle of related theorems which form the natural foundation for the theory of functions. Among the matters here treated are found the Weierstrass theorems upon series and an investigation of the properties of rational functions.

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* His cylinder 25a may, when fully deciphered, contribute something.

The eighth chapter is devoted to the theory of multiply valued functions, and gives a geometric treatment of Riemann surfaces, that is, a discussion employing the fruitful method of conformal depiction. With the ninth chapter, on analytic continuation, the foundations of the theory of functions reach a certain degree of completeness.

Next follow applications of the theory to periodic functions, a discussion of developments in series and in products, with a chapter on the elementary functions from the point of view of the general theory. The volume closes with an independent theory of the logarithmic potential, devoted to setting forth the view that the entire theory of functions could be developed upon this basis, without any reference to the preceding chapters. The theory of analytic functions of more than one variable, and the subject of definite integrals, could not be reached in the limits set for this first volume.

The infinitesimal calculus forms, as we see, together with a portion of the theory of point sets, the ground work for our analytic structure. There has been for many years no dearth of rigorous expositions of this part of analysis. In most of these, however, the theory of real functions appears as an end in itself, and consequently definitions and theorems are formulated in greater generality than is needful for the theory of complex functions, while the methods are left to be deciphered out of epsilon proofs. Here, on the contrary, it has been my principal aim to set forth the theory of complex functions in a form suited even for a first reading and joining closely to the calculus of infinitesimals. On this account I found it desirable to collect in the introductory chapters, in the simplest possible formulation, the fundamental propositions of the analysis of reals; and to explain with the extreme of clearness the methods of proof used in modern analysis. In Chapter 5 will be found certain more special investigations of point sets, indispensable for a satisfactory development of the subject.

To go into matters of detail, let me allude first to the geometric method of treating uniform convergence in the case of real functions (Chapter 3). By geometric intuition, employing curves and their related areas and tangential directions, we obtain a useful insight into the nature of the double limit process; and without an exact knowledge of this one can never attain to a thorough understanding of analysis.

In Chapter 7 I discuss the Weierstrass theorems on series by the aid of a theorem due to Morera, which allows a decided simplification of the proofs. The theorems themselves become more perspicuous by the discarding of what was nonessential, namely the frequent reference to power series; for in fact the most important of them relate primarily to *functions*, and it is here of no moment whatever that those functions happen to be developable by Taylor's theorem. Regarding power series indeed, it would be possible to go much further. It may not be generally known that the development in Taylor's series can be omitted completely from the fundamentals of the theory of functions; that indeed the proofs would be simpler if we were to use solely the analogue of the mean value theorem of differential calculus. On practical grounds, however, it is better not to banish that series entirely, for it serves to give the beginner practice in dealing with series in general.

On Riemann surfaces, Chapter 8 follows as a model Klein's presentation in his Leipzig lectures of 1881–82, making use also of a supplementary theorem due to Darboux (§5) on conformal depiction *in extenso*. To the theory of analytic continuation, in Chapter 9, I have devoted a more elaborate exposition than is usual, because it involves a throng of questions which must be settled with scrupulous care if the theory is to maintain at this point the same high level of rigor as elsewhere. For the same reason the developments of Chapter 5, §§ 3–10 (point sets) were requisite.

Among the earliest applications of the Cauchy theory were Liouville's lectures of 1847 upon doubly periodic functions. Even to this day indeed one can do no better, for the purpose of producing an adequate knowledge of the theory, than to discuss in detail this special class of functions.

It is doubtless the general custom to start out from arithmetic and geometry to explain the elementary functions. The theory of these functions can be rendered, however, simpler and more interesting if we lay down first the definition of the logarithm as an integral, and base upon this the powers and the exponential function. On the other hand, if one undertakes to treat trigonometric functions analytically, then the natural basis is given by the linear differential equation for the simplest case of vibration, simple harmonic motion. Neither of these plans is new; what I have attempted is their execution in a simple and systematic manner.

As regards the literature, let me refer to my Report: "General theory of analytic functions *a*) of one and *b*) of two or more complex quantities," in the Encyklopädie der mathematischen Wissenschaften, II, B1. In the present work I give few references to fundamental memoirs on the theory of functions, but have added references to special research papers not mentioned in the Encyklopädie article. These references lay no claim to completeness.

The arrangement of material is logically systematic. It is not, however, the only possible order for an introduction to function theory. For example, one might begin, as Klein has done, with Riemann surfaces, discuss briefly the Cauchy integral theorems, and then proceed to integrals upon Riemann surfaces (abelian integrals). Such an arrangement of material has this advantage, that the difficult parts of analysis are postponed, while the student begins with subjects more directly related to geometry than to analysis. A third arrangement, different from both, is found in French works, for example in Humbert's Cours d'analyse.

[The preface closes with acknowledgments to technical assistants and to the firm of Teubner.]

H. S. WHITE.

SHORTER NOTICES.

Arithmétique Graphique. Introduction a l'Etude des Fonctions Arithmétique. Par G. ARNOUX. Paris, Gauthier-Villars, 1906. xx + 225 pp.

THIS is the second of two volumes published by the author under the general title Arithmétique graphique. The first of these appeared in 1894 with the special title Les espaces arithmétiques hypermagiques. It has to do with magic squares, cubes, etc., but its methods are those of the theory of numbers. While it was still in press an abstract of it was presented to the Société mathématique de France by C. A. Laisant,[*] who was so impressed with its methods that he made the following prophecy: "Il y a en effet une telle originalité, une telle puissance d'invention dans les méthodes dont il s'agit, que je serais bien étonné si l'arithmétique n'arrivait pas quelque jour à les utiliser, soit pour obtenir des démonstrations plus simples de vérités connues déjà, soit pour découvrir des vérités nouvelles."

The starting point of Arnoux's method is the definition of a modular arithmetic space. A one-dimensional arithmetic space of modulus m consists of the m points of a straight line whose coordinates are $0, 1, 2, \cdots, m-1$. A two-dimensional arithmetic space consists of the m^2 points of a plane whose cartesian coordinates are the m^2 pairs of the integers $0, 1, 2, \cdots, m-1$. The extension to three or more dimensions is obvious. An arithmetic space of modulus m may be thought of as a *finite geometry* that corresponds to the *finite algebra* of integers modulo m. An arithmetic straight line in a two-dimensional arithmetic space consists of the m points whose coordinates satisfy a linear congruence of the form $ax + by + c \equiv 0 \pmod{m}$.[†] The points of such a line do not in general all lie on a euclidean straight line but there is a close connection between the two. For those points of a euclidean plane whose coordinates are integers may be arranged in an infinite number of arithmetic spaces of modulus m by reducing the coordinates modulo m. A euclidean straight line determined by two points A and B of one of these modular arithmetic spaces will cross an infinite number of other such arithmetic spaces, but it will contain only m points that have different pairs of coordinates. The points of the given arithmetic space that have these m different pairs of coordinates are the m points of the arithmetic straight line determined by A and B. An arithmetic plane in a modular arithmetic space of three dimensions consists of the $m(m+1)$ points whose coordinates satisfy a congruence of the form $ax + by + cz + d \equiv 0 \pmod{m}$. An arithmetic line in such a space consists of the points common to two arithmetic planes.

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[*] *Bulletin de la Soc. Math. de France*, vol. 22, pp. 28–36.

[†] Arnoux gives an equivalent definition in terms of the function $ax + by$, and the modulus m.

The most interesting arithmetic spaces are those in which the modulus is a prime number p, for in the finite algebra of integers modulo p the formal laws of addition, subtraction, multiplication and division are as in ordinary algebra. Arnoux's use of these finite geometries for the construction of magic squares is quite original and well worth reading.

The book under review is an attempt to illuminate the theory of arithmetic functions by the use of modular arithmetic spaces which furnish a means of graphical representation and make it possible to use the convenient language of geometry, but in the reviewer's opinion it fails to fulfill the prophecy of Laisant quoted above.

The book is very well described by its author in the first sentence of the preface in these words: "Ce livre n'est point un traité didactique sur les fonctions arithmétiques, mais plutôt une simple causerie sur ce sujet si intéressant et si peu connu." This precludes many criticisms that might otherwise be made, for it is not a good book for a beginner in the subject although it is described in the title as an introduction to the theory of arithmetic functions. The method throughout is largely synthetic and inductive. The many tables and the special cases used in leading up to the proofs of theorems make the number of pages large for the subject matter treated. The use of modular arithmetic spaces is interesting but not so effective as it was in the author's earlier work on magic squares in which these finite geometries played such an important part.

W. H. BUSSEY.

Studien über die Methoden von C. Neumann und G. Robin zur Lösung der beiden Randwertaufgaben der Potentialtheorie. Von ERNST RICHARD NEUMANN. Leipzig, Teubner, 1905.

This work was submitted to the Jablonowski Gesellschaft of Leipzig in 1902 in response to the wish of the society "that the investigations of Poincaré's memoir entitled 'La méthode de Neumann et le problème de Dirichlet,' 1896, be in some particular essentially completed," and was awarded the prize offered. The author sets himself the task of establishing the convergence of the series obtained by C. Neumann and Robin for a region bounded by a surface which is not necessarily convex and for boundary values of which continuity alone is demanded. In doing so, however, his first object is complete rigor and uniformity of method of proof. The method employed is based upon the integrals used by Poincaré of the type

$$\int\left[\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y}+\frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial z}\right]d\tau$$

applied to the successive potentials of C. Neumann and Robin and also to the "polar potentials" introduced by the author in a note in the *Göttinger Nachrichten*, 1899, page 291. These polar potentials are simply the successive Neumann and Robin potentials that would arise in forming the Green functions for the two corresponding problems, and their introduction goes far toward unifying and simplifying the discussion. Using them, together with his two theorems of the "constancy of the moment" (see pages 50–51, also *Mathematische Annalen*, volume 54, page 40), the author succeeds in establishing the limits which the terms, or more strictly in the case of the Neumann series, pairs of terms of his series approach (pages 112–114), but quite without establishing any criteria for the rapidity of approach necessary for a convergence proof. To attain this he finds it necessary to make use of a "Poincaré principle," namely that if U be the potential of any simple distribution on the boundary surface σ, the total mass being 0, then the ratio of the value of the integral

$$\int\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial U}{\partial y}\right)^2+\left(\frac{\partial U}{\partial z}\right)^2\right]d\tau$$

when extended over the space outside σ to its value when extended over the space within σ lies between two constants greater than 0. All the usual results with respect to the series follow immediately. But uniformity and rigor have been sacrificed, for it is explicitly stated that the question of the proof of the above theorem is left open. In Poincaré's work it is derived on the basis of certain "Poincaré transformations," without appeal to Dirichlet's principle, and E. R. Neumann's work deserves credit for crystallizing the difficulty of the situation in this one theorem, particularly in view of the fact that for a large class of surfaces it has been proven (Korn: Lehrbuch der Potentialtheorie, I, pages 241 and 245), and further generalization is probably possible.

The results desired by the Jablonowski Gesellschaft have on the other hand since been attained in a more far reaching manner through Fredholm's work, especially in its application by J. Plemelj (*Monatshefte für Mathematik und Physik*, volume 15, 1904). The matter of convergence is completely settled and interest in the work of the more immediate followers of Poincaré becomes historical in its nature, with the important exception of certain studies of the behavoir with respect to continuity of surface distributions and their derivatives, upon which the Fredholm method for the potential problems also depends. The present work is of value in both respects and is moreover to be commended for its style and arrangement.

O. D. KELLOGG.

Vorlesungen über mathematische Näherungsmethoden. By Dr. OTTO BIERMANN. Braunschweig, Friedrich Vieweg und Sohn, 1905. 227 pp.

"In recent years much has been done to meet the requirements of those who find it necessary to make use of mathematical methods, and it is therefore surprising that as yet no book exists which treats of methods of approximation in mathematics in clear and concise form and so as not to require much preliminary mathematical knowledge." This, the first sentence of the preface, indicates clearly the object with which the book under review was written. We may say at once that it fulfills this object admirably.

The book is divided into six parts treating respectively of: I, calculation with exact and approximate numbers; II, numerical computation in higher analysis; III, the approximate solution of equations; IV, interpolation and the calculus of finite differences; V, application of the methods of interpolation to approximate quadrature and cubature; VI, certain mathematical instruments. It is indeed a startling fact that until now there should be no book devoted to a connected exposition of these topics. Much of the material here collected can of course be found elsewhere. The use of infinite series in numerical calculations (II) is discussed in works on higher algebra and in books devoted exclusively to such series; graphical methods for solving equations (V) are treated in many books on algebra and the theory of equations; we have treatises, large and small, on the calculus of finite differences and methods of interpolation; and approximate methods of integration are discussed more or less adequately in many of the standard texts on the calculus. But that a connected treatment of such topics is desirable needs no argument.

Now that such a treatment exists, however, an argument in the opposite direction presents itself forcefully: we need more of the material of this little treatise in our elementary textbooks. Much of what is contained in part I, on computation with approximate numbers — obtained from measurement, or from exact numbers by approximate processes of calculation — should be contained in our texts on trigonometry, where the student first meets seriously the problem of numerical computation with approximate data. It is manifestly absurd to carry out the computation of the hypotenuse of a right-angled triangle to three decimal places when the two sides have been measured only to within the nearest tenth. Yet such things are done continually in the majority of our present texts. We believe also that the graphical and other approximate solutions of equations and approximate methods of integration should receive more adequate treatment than is usually accorded them in our elementary courses. If the answer is made that there is no time for such things, we would suggest that it might be worth while either to sacrifice some of the topics of more special theoretical interest, or to establish separate short courses on "approximate methods" covering some such ground as does the text under review.

Returning to the latter, the combination of clearness and simplicity of presentation with rigor of treatment is admirably done. The stickler for mathematical precision will find little to criticise, while the seeker for "practical" information will find the text unencumbered with logical subtleties. Much of the contents, moreover, is of recent date, as is evidenced by the numerous references to journals.

J. W. YOUNG.

La Statique Chimique Basée sur les deux Principes Fondamentaux de la Thermodynamique. Par E. ARIÈS. Paris, A. Hermann, 1904. viii + 251 pp.

THAT the investigations of Willard Gibbs on the purely theoretical side of physical chemistry have taken a vital place in recent French researches whether on the theoretical or practical side of the subject is due partly to the seed sown long ago by the pioneer Massieu and later assiduously cultivated by Duhem, partly to the translations of Gibbs's memoirs by Le Chatelier and Brunhes, and partly no doubt to the Frenchman's natural admiration and love for a logically developed theory. Ariès in his work on chemical statics shows traces of Massieu, of Duhem, and particularly of Gibbs. In fact, although his notation is not at all that of Gibbs, his ideas are by his own admission largely taken from him.

The book under review is an excellent account of the more important results of the theory of chemical equilibrium. As such books are not numerous and as the original memoirs of Gibbs are not always easy reading, Ariès's volume cannot help but be a very useful addition to the literature of the subject. After presenting the fundamentals of the theory, the author takes up in separate chapters several different applications. We may mention: Change of state and analogous phenomena, some types of dissociation, solutions, monovariant systems, the separation of mixed liquids, mixed gases in equilibrium with mixed liquids, perfect gases, the law of Dalton and Gibbs's principle, dilute solutions, and osmosis.

From this partial list of the subjects treated it will appear that we have here a tolerably complete and systematic account of those parts of theoretical physical chemistry which are likely to be of greatest use to the reader. It is well to mention that in many instances the illustrations and discussions of the author may serve as elucidations of parts of Gibbs's work which on account of their very generality are difficult of comprehension. In particular we should like to call attention to the theory of gaseous equilibria and the principle of Gibbs, chapters XII and

XIII. The statement of the principle of Gibbs and its bearing on the behavior of gaseous mixtures is treated with exceptional clearness, and the fundamental fact that the theory of such mixtures depends on only three laws (those of Mariotte, Joule, and Gibbs) is so emphasized that no reader can miss it or fail to see its significance.

E. B. WILSON.

NOTES.

THE opening (January) number of volume 29 of the *American Journal of Mathematics* contains the following papers: "The groups which contain less than fifteen operators of order two," by G. A. MILLER; "Concerning the improper definite integral," by N. J. LENNES; "On the congruence of axes in a bundle of linear line complexes," by O. P. AKERS; "On septic scrolls having a rectilinear directrix," by C. H. SISAM. This number also contains a portrait of Professor DAVID HILBERT, of the University of Göttingen.

The April number contains: "Beiträge zur nicht-euklidischen Geometrie, I, II, III," by E. STUDY; "Certain triply orthogonal systems of surfaces," by L. P. EISENHART.

THE April number (volume 8, number 3) of the *Annals of Mathematics* contains: "Note on the use of group theory in elementary trigonometry," by G. A. MILLER; "The asymptotic lines on the anchor ring," by M. B. WHITE; "On certain theorems of mean value for analytic functions of a complex variable," by D. R. CURTISS; "Note on regular polygons," by C. A. SCOTT; "The revolution of a dark particle about a luminous center," by E. B. WILSON.

AT the meeting of the London mathematical society, held on March 14, the following papers were read: By G. W. EVANS-CROSS, "Exhibition of a new calculating machine"; by T. STUART, "On the reduction of the factorization of binary septans and octans to the solution of indeterminate equations of the second degree"; by L. E. DICKSON, "Invariants of the general quadratic form modulo 2"; by J. BRILL, "On partial differential equations of the first order."

THE regular spring meeting of the Association of teachers of mathematics in the Middle States and Maryland was held at Teachers College, Columbia University, on Saturday, April 6, 1907. The following papers were presented: By W. H. MALTBIE, "Zero and infinity"; by GEORGE H. HALLETT, "Some problems in the teaching of elementary mathematics"; by D. E. SMITH, "The history of computation," with stereopticon illustrations; by W. E. STARK, "The use of certain ancient instruments in the modern teaching of geometry and trigonometry," with stereopticon illustrations.

AT the meeting of the Rochester section of the Association of teachers of mathematics in the Middle States and Maryland, held at the University of Rochester on April 13, 1907, the following papers were read: By W. P. DURFEE, "Elementary algebra"; by M. M. WARDWELL, "Elementary algebra as a preparation for the study of algebra"; by E. D. GRABER, "Graphical representations in algebra"; by C. C. WILCOX: "The method of limits."

THE list of members of the Deutsche Mathematiker Vereinigung, published in the January number of the *Jahresbericht*, contains 690 names, including 63 Americans; the financial report shows a credit of nearly 20,000 marks. The Vereinigung is now completing its seventeenth year, having been founded in 1890 at the Bremen meeting of the society of German naturalists and physicians. It immediately began the publication of the *Jahresbericht*, which contained each year in addition to the list of members, the financial statement, and an abstract of the papers read at the annual meeting, a comprehensive report of some branch of mathematics. Twelve well-known reports have now appeared in ten volumes of the first series, and another will soon be issued. In 1893 the Vereinigung prepared the extensive exhibition of models and apparatus at Munich, also shown later in Chicago at the Columbian exposition, and in 1894 it definitely formulated the plan of the Encyklopädie der mathematischen Wissenschaften. Upon the appearance of the earlier parts of this work, the preparation of further yearly reports was discontinued, the same ends being attained by the Encyklopädie. In 1902 the *Jahresbericht* began to appear as a monthly publication, somewhat similar to the BULLETIN. In 1904 the Vereinigung primarily conducted the third international congress of mathematicians at Heidelberg, and published its proceedings. During the last few years it has also been influential in bringing about important changes in the programme of the gymnasia and other preparatory schools. At the forthcoming meeting, to be held in Dresden, September 15–22, further questions of pedagogy will be considered and plans perfected for providing a suitable place for its extensive library and especially for preserving valuable manuscripts. The officers for the present year are: president, Professor A. VON BRILL; secretary, Professor A. KRAZER; editor of the *Jahresbericht*, Professor A. GUTZMER.

THE educational museum of Teachers College of Columbia University announces the preparation of nearly 200 lantern slides of illustrations useful in teaching the history of elementary mathematics. The slides will be furnished only to schools and colleges, or to those who give courses in such institutions, at cost, which is forty cents per slide. No order for less than twenty-five slides can be filled. For the list of subjects and all other information concerning them, letters should be addressed to the Educational Museum, Teachers College, Columbia University, New York City.

UNDER the title of *Revue trimestrielle de Mathématiques pures et appliquées* a new mathematical publication is about to appear, edited by E. Weber, Liège, Belgium. Its pages are open to meritorious papers in pure mathematics, hypergeometry, mechanics, and elasticity. The price per volume of 300 pages is 15 francs.

THE scientific society of Harlem announces the following prize problem for 1908: "To determine how p_1N spheres of radius R_1 and p_2N spheres of radius R_2 (N being undetermined) should be placed to occupy the least space? When p_1, p_2 are known, what are the critical relations between R_1, R_2 such that by making a slight variation of this relation an entirely different distribution of the spheres results? Partial results, limited solutions, particular cases, etc., will be considered. Competing memoirs should be sent to Dr. J. Bosscha, Harlem, Holland, under the usual conditions, before January 1, 1908. The value of the prize is 500 gulden. The winner may choose between the cash value and a gold medal.

THE subject of the next Adam's prize problem, to be awarded in 1909, is "To discuss the radiation from electric systems or ions in accelerated motion and the mechanical relations on their motion which arise from it." Competition is open to anyone who has taken a degree at Cambridge University. Memoirs should be submitted to the vice-chancellor before December 16, 1908.

AMONG the Smith prizes for the present year, the two following were awarded for mathematical essays: To A. S. EDDINGTON, "The systematic motions of the stars," and to J. W. NICHOLSON, for his essay "The bending of waves around a large opaque sphere and some associated problems."

DURING the year 1906 the following doctorates with mathematics as major subject were conferred by the University of Paris: (Doctoral d'Etat.) CARRUS, "Familles de surfaces à trajectoires orthogonales"; M. FRÉCHET, "Sur quelques points du calcul fonctionnel"; LATTÈS, "Sur les équations fonctionnelles"; FAYET, "Recherches concernant les excentricités des comètes"; SAINT BLANCAT, "Action d'une masse intramercurielle sur la longitude de la lune" — (Doctorat d'Université.) REMOUNDOS, "Sur les zéros d'une classe de fonctions transcendantes."

THE following courses in mathematics are announced for the summer sessions of the several universities:

UNIVERSITY OF CHICAGO (summer quarter, June 15 to September 15). — By Professor E. H. MOORE: Graphical methods in algebra especially for teachers, five hours; Theory of determinants, advanced course, four hours; Seminar, two hours. — By Professor O. BOLZA: Theory of functions of a complex variable, four hours; Problems in the theory of functions, two hours; Abelian functions, two hours. — By Professor H. E.. SLAUGHT: Integral calculus, five hours; Differential equations, four hours. — By Professor L. E. DICKSON: Trigonometry, five hours; Solid analytic geometry, five hours; Continuous groups, four hours. — By Professor F. R. MOULTON: Descriptive astronomy, five hours; General celestial mechanics, five hours. — By Professor K. LAVES: Descriptive astronomy, five hours; General astronomy with laboratory work, five hours. — By Dr. A. C. LUNN: Curve tracing and differential calculus, five hours; dynamics of oscillatory systems, four hours. — By Mr. N. J. LENNES: Plane analytic geometry, five hours;

Critical review of secondary mathematics, four hours. — By Professor G. W. Myers: Pedagogy of elementary school mathematics, five hours; Pedagogy of secondary school mathematics, five hours.

Columbia University (summer session July 9 to August 17). — Py Professor James Maclay: Differential calculus, five hours; Differential geometry, five hours. — By Professor C. J. Keyser: Plane analytic geometry, five hours; Modern analytic methods in geometry, five hours. — By Professor Edward Kasner: Differential equations, five hours. — By Dr. G. H. Ling: Integral calculus, five hours; Theory of functions of a complex variable, five hours. — By Mr. Maxson: Analytic geometry, five hours.

University of Pennsylvania (summer session, July 8 to August 17). — By Professor E. S. Crawley: Higher analytic geometry, five hours. — By Professor I. J. Schwatt: Definite integrals, five hours. — By Professor G. H. Hallett: Theory of functions of a complex variable, five hours. — By Dr. F. H. Safford: Differential equations, five hours.

The following courses in mathematics will be offered during the year 1907–1908.

Yale University. — By Professor J. Pierpont: Introduction to the theory of functions, two hours; Projective geometry, two hours; Elasticity and hydromechanics, two hours; Elliptic functions, two hours. — By Professor P. F. Smith: Higher geometry, two hours; Geometric analysis, one hour. — By Professor E. W. Brown: Mechanics, two hours; Celestial mechanics, two hours. — By Professor H. E. Hawkes: Algebra and analytic geometry, two. hours; Teachers' course in geometry, two hours; Advanced algebra, two hours. — By Professor M. Mason: Differential equations, two hours; Integral equations, one hour; conformal mapping and Riemann surfaces, one hour. — By Professor E. B. Wilson: Molecular properties of matter, two hours; Gravitation and electrostatics, one hour. — By Dr. W. A. Granville; Differential geometry, two hours. — By Dr. L. I. Hewes: Differential equations, one hour; geometric transformations of the plane and of space, two hours; Graphical and numerical computation, one hour. — By Dr. W. R. Longley; Differential geometry, two hours. The mathematical club meets fortnightly.

THE following courses in mathematics are announced by the various foreign universities for the summer semester:

UNIVERSITY OF BERLIN. — By Professor H. A. SCHWARZ: Integral calculus, with exercises, four hours; Theory of elliptic functions and selected chapters of the theory of functions, four hours; Seminar, two hours; Colloquium, two hours. — By Professor G. FROBENIUS: Theory of algebraic equations, four hours; Seminar, two hours. — By Professor F. SCHOTTKY: Theory of potential, four hours; Special theory of functions, two hours; Seminar, two hours. — By Professor G. HETTNER: Fourier series and integrals, four hours. — By Professor J. KNOBLAUCH: The influence of Euler's work on modern mathematics, two hours; Theory of surfaces, II, four hours; Theory of space curves, two hours. — By Professor R. LEHMANN FILHÈS: Differential calculus with exercises, four hours. — By Professor E. LANDAU: Calculus of variations, four hours; Theory of probabilities, two hours. — By Dr. I. SCHUR: Theory of determinants, four hours.

UNIVERSITY OF BONN. — By Professor E. STUDY: Differential geometry, four hours; Selected chapters in mechanics, two hours; Seminar, two hours. — By Professor G. KOWALEWSKI: The division of the circle, four hours; General theory of differential equations, four hours. — By Professor F. LONDON: Elements of differential and integral calculus, four hours; with exercises, two hours; Axonometry and perspective, two hours. — By Dr. E. SCHMIDT: Definite integrals, four hours; Foundations of point sets and the theory of real functions, two hours.

UNIVERSITY OF GÖTTINGEN. — By Professor F. KLEIN: Curves and surfaces, four hours; Seminar, two hours. — By Professor D. HILBERT: Theory of differential equations of one variable, four hours; Seminar, two hours. — By Professor H. MINKOWSKI: Calculus of variations, four hours; Radiation, two hours; Seminar, two hours. — By Professor C. RUNGE: Numerical solution of equations, with exercises, four hours; Photogrammetry, with exercises, two hours; Seminar, two hours. — By Professor L. BRENDEL: Insurance, four hours; Seminar, two hours. — By Professor L. PRANDTL: Introduction to thermodynamics, four hours; Theory of machines, two hours; Seminar, two hours. — By Dr. F. HERGLOTZ: Linear differential equations of complex variables, four hours. — By Dr. C. CAR-

ATHÉODORY: Differential and integral calculus, four hours. — By ——————: Analytic geometry, four hours. — By Dr. M. ABRAHAM: Hydrodynamics, four hours.

UNIVERSITY OF LEIPZIG: By Professor C. NEUMANN: Selected chapters of mathematics, four hours. — By Professor A. MAYER: Analytic dynamics, with exercises, four hours. — By Professor O. HÖLDER: Theory of functions of a complex variable, four hours; Definite integrals, four hours; Seminar, two hours. — By Professor K. ROHN: Projective geometry, four hours; Plane analytic geometry, four hours; Seminar, two hours. — By Professor A. VON OETTINGEN: Geometric drawing, four hours.— By Professor F. HAUSDORFF: Algebraic equations, four hours. — By Professor H. LIEBMANN: Ordinary differential equations, with exercises, four hours.

UNIVERSITY OF OXFORD (*Easter and Trinity terms*). — By Professor W. ESSON: Comparison of analytic and synthetic methods in the theory of conics, two hours; Informal instruction in geometry, one hour. — By Professor E. B. ELLIOTT: Theory of functions, three hours. — By Professor A. E. H. LOVE: Waves and sound, three hours. — By Mr. A. L. DIXON: Calculus of variations, one hour. — By Mr. H. T. GERRANS: Line geometry, two hours. — By Mr. A. E. JOLLIFFE: Invariants and covariants of conics, one hour. — By Mr. P. J. KIRKBY: Higher plane curves, three hours. — By Mr. J. W. RUSSELL: Rigid dynamics, two hours. — By Mr. E. H. HAYES: Electrostatics, one hour. — By Mr. R. F. MCNEILE: Algebra, two hours. — By Mr. C. E. HASELFOOT: Series and continued fractions, two hours. — By Mr. A. L. PEDDER: Spherical trigonometry, one hour. — By Mr. C. H. SAMPSON: Solid geometry, two hours. — By Mr. C. H. THOMPSON: Differential equations, two hours.

UNIVERSITY OF PARIS (March 1 to June 15). — By Professor E. PICARD: Integrals of partial differential equations, two hours. — By Professor E. GOURSAT: Differential equations, one hour. — By Professor L. RAFFY: Theory of curvature of lines traced on surfaces, one hour; Conference, two hours. — By Professor P. PAINLEVÉ: Kinematics, two hours. — By Professor P. APPELL: Analytic mechanics, three hours. By Professor E. BOREL: Calculus of probabilities, one hour.— (In the Ecole Normale) — By Professor J. TANNERY: Differential and integral calculus, two hours.—By Professor L. RAFFY: Applications of analysis to geometry, two hours.—By Professor E. BOREL, and by Professor J. HADAMARD: Courses in general mathematics, two hours.

UNIVERSITY OF STRASSBURG.—By Professor T. REYE: Selected chapters of higher synthetic geometry, four hours; Theory of potential, two hours; Seminar, two hours.—By Professor H. WEBER: Definite integrals and introduction to the theory of functions, four hours; Algebra, four hours; Seminar, two hours.—By Professor M. SIMON: Teaching of mathematics in schools, four hours.—By Professor J. WELLSTEIN: Introduction to the theory of algebraic functions, four hours; Riemann surfaces, two hours; Seminar, two hours.—By Professor H. E. TIMERDING: Analytic geometry of space, four hours; Descriptive geometry, II, with exercises, four hours; Development of mechanical principles, two hours; Seminar, two hours.—By Dr. P. EPSTEIN: Analytic theory of numbers, four hours.

THE University of Glasgow has conferred the honorary degree of doctor of laws on Professor H. POINCARÉ, of the University of Paris, and on Professor U. DINI, of the University of Pisa.

AT the University of Chicago Professor H. MASCHKE has been promoted to a full professorship of mathematics, and Professor L. E. DICKSON to an associate professorship of mathematics.

PROFESSOR E. B. WILSON, of Yale University, has been appointed to an associate professorship of mathematics at the Massachusetts Institute of Technology.

AT Haverford College, Professor L. W. REID has been promoted to a full professorship of mathematics.

PROFESSOR J. B. WEBB, of the Stevens Institute of Technology, will retire from the active duties of his professorship at the close of the present academic year.

MR. E. A. MORITZ has resigned his instructorship at the University of Wisconsin to accept a business position in electric engineering.

THE deaths are announced of Professor A. SUCHARDA, of the technical school at Brünn, on February 20, at the age of 52 years; Dr. J. LYON, of the University of Geneva, on January 26, at the age of 49 years; Professor G. G. TSWETKOFF, of the Moscow school of forestry.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AHRENS (W.). C. G. J. Jacobi als Politiker. Beitrag zu seiner Biographie. Leipzig, Teubner, 1907. 8vo. 45 pp. M. 1.20

——. See JACOBI (C. G. J.).

BERTINI (E.). Introduzione alla geometria proiettiva degli iperspazi, con appendice sulle curve algebriche e loro singolarità. Pisa, 1906. 8vo. 426 pp. L. 15.00

BOHLIN (K.). Zur Theorie der algebraischen Gleichungen. Upsala, 1906. 8vo. 55 pp.

BOLZANO (B.). Rein analytischer Beweis, dass zwischen je zwei Werten, die ein entgegengesetztes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege. — HANKEL (H.). Untersuchungen uber die unendlich oft oszillirenden und unstetigen Functionen. — Herausgegeben von P. E. B. Jourdain. Leipzig, 1906. 8vo. 115 pp. Cloth. M. 1.80

BOPP (K.). Die Kegelschnitte des Gregorius a St. Vincentio in vergleichender Bearbeitung. (Habilitationsschrift.) Heidelberg, 1906. 8vo. 42 pp.

BÜRKLEN (O. T.). Formelsammlung und Repetitorium der Mathematik, enthaltend die wichtigsten Formeln und Lehrsätze der Arithmetik, Algebra, algebraischen Analysis, ebenen Geometrie, Stereometrie, ebenen und sphärischen Trigonometrie, mathematischen Geographie, analytischen Geometrie der Ebene und des Raumes, der Differential- und Integralrechnung. 3te, durchgesehene Auflage; 2ter Abdruck. Leipzig, 1906. 8vo. 227 pp. M. 0.80

CHANDLER (G. H.). Elements of the infinitesimal calculus. 3d edition. New York, Wiley, 1907. 12mo. 6 + 319 pp. Cloth. $2.00

FOURREY (E.). Curiosités géométriques. Paris, Vuibert et Nony, 1907. 8vo. 8 + 431 pp. Fr. 3.50

GANTER (H.) und RUDIO (F.). Elemente der analytischen Geometrie. 6te, verbesserte Auflage. (In 2 Teilen.) Teil I: Analytische Geometrie der Ebene. Leipzig, 1906. 8vo. 8 + 190 pp. Cloth. M. 3.00

GAUSS (C. F.). Werke, herausgegeben von der k. Gesellschaft der Wissenschaften zu Göttingen. Vol. VII: Theoria motus und theoretisch-astronomischer Nachlass (parabolische Bewegung, Störungen der Ceres und der Pallas, Theorie des Mondes). [Redigiert von M. Brendel.] Leipzig, Teubner, 1906. 4to. 650 pp. Boards. M. 30.00

GEIGENMÜLLER (R.). Leitfaden und Aufgabensammlung zur höheren Mathematik; für technische Lehranstalten und den Selbstunterricht bearbeitet. Vol. I: Die analytische Geometrie der Ebene und die algebraische Analysis. 7te Auflage. Mittweida, Polytechnische Buchhandlung, 1907. 8vo. 12 + 290 pp. Cloth. M. 6.00

GUNDELFINGER (S.). See Hesse (O.).

HANKEL (H.). See BOLZANO (B.).

HESSE (O.). Vorlesungen aus der analytischen Geometrie der geraden Linie, des Punktes und des Kreises in der Ebene. 4te Auflage, revidiert und ergänzt von S. Gundelfinger. Leipzig, Teubner, 1906. 8vo. 8+251 pp. Cloth. M. 6.00

HÖFLER (A.). See KLEIN (F.).

JACOBI (C. G. J.). Neue Methode zur Integration partieller Differentialgleichungen erster Ordnung zwischen irgend einer Anzahl von Veränderlichen. Herausgegeben von G. Kowalewski. Leipzig, Engelmann, 1906. 12mo. 228 pp. Cloth. (Ostwald's Klassiker der exakten Wissenschaften, No. 156.) M. 4.00

JACOBI (C. G. J.) und JACOBI (M. H.). Briefwechsel; herausgegeben von W. Ahrens. Leipzig, Teubner, 1907. 8vo. 20 + 282 pp., 2 portraits. (Abhandlungen zur Geschichte der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Heft 22.) M. 6.90

JOURDAIN (P. E. B.). See BOLZANO (B.).

KLEIN (F.) und HÖFLER (A.). Grenzfragen der Mathematik und Philosophie. Vorträge, gehalten am 14. Oktober 1905 und 19. Januar 1906 in der Philosophischen Gesellschaft an der Universitat zu Wien. Leipzig, Barth, 1906. 8vo.

KOWALEWSKI (G.). See JACOBI (C. G. J.).

LENNES (N. J.). See VEBLEN (O.).

MÜNICH (K.). Ueber nichteuklidische Cykliden. (Diss.) München, 1906. 8vo.

NELSON (L.). Kant und die nicht-euklidische Geometrie. Berlin, Treptow-Sternwarte, 1906. 8vo. 28 pp. (Vorträge und Abhandlungen, herausgegeben von der Zeitschrift *Das Weltall* unter Leitung von F. S. Archenhold, Heft 13.) M. 1.50

OSBORNE (G. A.). Differential and integral calculus, with examples and applications. Revised edition. Boston, Heath, 1906. 8vo. 400 pp. Half leather. $2.00

OSGOOD (W. F.). Lehrbuch der Functionentheorie. (In 2 Bänden.) Vol. I, 2te Hälfte. Leipzig, Teubner, 1907. 8vo. 12 pp. and pp. 307–642. (B. G. Teubners Sammlung von Lehrbuchern auf dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Vol. XX : 1.) M. 7.60

PETIT BOIS (G.). Tafeln unbestimmter Integrale. Leipzig, Teubner, 1906. 8vo. 12 + 154 pp. M. 8.00

RUDIO (F.). See GANTER (H.).

SAUER (K.). Zur Funktionentheorie auf dem algebraischen Gebilde $S=\sqrt[3]{f_{3n}(z)}$. (Diss., Strassburg.) Leipzig, Teubner, 1906. 8vo.

SAVIS (P.). Sopra le congruenze. Parma, Ferrari, 1906. 4to. 12 pp.

TAYLOR (F. G.). An introduction to the differential and integral calculus and differential equations. 2d edition. New York, Longmans, 1906. 12mo. 24 + 568 pp. Cloth. $3.00

VEBLEN (O.) and LENNES (N. J.). Introduction to infinitesimal analysis; functions of one real variable. New York, Wiley, 1907. 8vo. 7 + 227 pp. Cloth. $2.00

ZAALBERG (A. L.). Differentiaal-meetkundige eigenschappen van stralenstelsels. Leiden, 1905. 8vo. 113 pp. M. 5.00

II. ELEMENTARY MATHEMATICS.

Academia de ingenieros del Ejército. Papeletas de aritmética, álgebra, geometría y trigonometría, para los exámenes de ingreso. Guadalajara, Concha, 1906. 8vo. 60 pp. P. 1.00

Baker (W.) and Bourne (A. A.). First geometry, with answers. London, Bell, 1907. 12mo. 142 pp. Cloth. 1s. 6d.

Bardey (E.). Aufgabensammlung, für bayerische Mittelschulen nach der neuesten Auflage bearbeitet von J. Lengauer. Leipzig, Teubner, 1907. 8vo. 4 + 192 pp. Cloth. M. 2.20

Bassi (A.). Risoluzione dei triangoli piani; norme ed esempi. Torino, Paravia, 1907. 8vo. 7 + 355 pp. (Biblioteca di scienze fisiche, matematiche e naturali, collezione Paravia.) L. 4.00

Bourne (A. A.). See Baker (W.).

Burali-Forti (C.). Lezioni di aritmetica pratica, ad uso delle scuole secondarie inferiori. 4ta edizione, accresciuta e riordinata. Torino, Gallizio, 1907. 8vo. 8 + 278 pp., 2 plates. L. 2.25

Diet (L.). Ueber die Kongruenz und das Kongruenzgefühl und uber die graphische Darstellung körperlicher Objekte. Wien, Schroll, 1907. 8vo. 18 pp. M. 1.20

Fawdry (R. C.). See Playne (H. C.).

Halsted (G. B.). Rational geometry. A text-book for the science of space (based on Hilbert's foundations). 2nd edition, thoroughly revised. New York, Wiley, 1907. 12mo. 8 + 273 pp. Cloth. $1.50

Handbuch fur Lehrer höherer Schulen. Bearbeitet von A. Auler, O. Borner, W. Capitaine, R. Fricke, E. Grimsehl, K. Jansen, F. Kuhlmann, F. Lampe, B. Landsberg, O. Lyon, H. Müller, J. Nelson, A. Rausch, B. Schmidt, E. Stiehler, H. Vollmer, E. Weede, O. Weissenfels, E. Wernicke, J. Ziehen. Leipzig, Teubner, 1906. 8vo.

Koch (W.). Zur Methode im geometrischen Unterricht. (Progr.). Sorau, 1906. 4to. 6pp., 1 plate. M. 1.20

Kraus (K.). Grundriss der Arithmetik für Lehrer- und Lehrerinnen-Bildungsanstalten. 3te Auflage; unveränderter Abdruck der 2ten Auflage. Wien, Pichler, 1906. 8vo. 316 pp. Cloth. M. 3.50

Lengauer (J.). See Bardey (E.).

Lorenzo y Aleu (M.). Pizarros de aritmética y álgebra, para uso de los aspirantes á ingreso en las academias militares y de la armada, dispuestas con sujeción á las obras declaradas de texto. 2a edición. Quaderno primero: Aritmética. Valdemoro, Guardia Civil, 1906. 8vo. 95 pp. P. 3.00

Model answers in matriculation mathematics, together with the examination papers from September 1902 to January 1907. London, Clive, 1907. 12mo. 188 pp. (University tutorial series.) 2s.

Nouvelles tables de logarithmes à cinq décimales pour les lignes trigonométriques dans les deux systèmes de la division centésimale et de la division sexagésimale du quadrant et pour les nombres de 1 à 12,000, suivies des mêmes tables à quatre décimales et de diverses tables et formules usuelles. 2e édition, revue et corrigée. Paris, Gauthier-Villars, 1906. 8vo. F. 4.50

Playne (H. C.) and Fawdry (R. C.). Practical trigonometry. London, Arnold, 1907. 12mo. 304 pp. Cloth. 6s.

Rubio (J. M.). Apuntes de geometría, con arreglo al programa exigido para el ingreso en el Cuerpo auxiliar de telégrafos. Madrid, Díaz, 1906. 12mo. 153 pp. P. 3.00

Soulé (G.). Intermediate philosophic arithmetic. New Orleans, 1906. 8vo. 9 + 589 pp. Cloth. $2.50

Tanner (J. H.). Elementary algebra. New York, American Book Company, 1907. 12mo. 364 pp. Cloth. (Modern mathematical series.) $1.00

Visscher (N. J.). Tal van oplossingen van rekenkundige vraagstukken, opgegeven bij acte-examen 1905 in de 1. inspectie. Groningen, 1906. 8vo. 116 pp. M. 1.50

Walsh (J. H.). Algebra and geometry for the Philippine schools, being chapters VII and VIII of Walsh's New grammar school arithmetic. Boston, Heath, 1906. 12mo. 3 + 88 pp. Cloth. $0.30

Wells (W.). Key to "Text-book in algebra." Boston, Heath, 1906. 12mo. 644 pp. Half leather. $2.00

Wright (D. S.). Exercises in concrete geometry, with supplementary discussions. Boston, Heath, 1906. 12mo. 4 + 84 pp. Cloth. $0.35

Young (E.). First year's course in geometry and physics. Parts I and II. London, Bell, 1907. 12mo. 110 pp. Cloth. 1s. 6d.

Young (J. W. A.). The teaching of mathematics in the elementary and the secondary school. New York, Longmans, 1907. 8vo. 18 + 351 pp. Cloth. (American teachers series, edited by J. E. Russell.) $1.50

III. APPLIED MATHEMATICS.

Coiffard (H.). Cours de mécanique et machines professé à l'école nationale d'arts et métiers. Statique et dynamique du point; statique des systèmes; statique graphique; hydraulique; roues et turbines. Lille, Danel, [1907]. 4to. 328 pp.

Correale (E.). Elementi di idrostatica e di idrodinamica per gli allievi ingegneri. Napoli, 1906. 8vo. 55 + 25 pp.

Creighton (W. H. P.). The steam engine and other heat motors. New York, Wiley, 1907. 8vo. 12 + 499 pp. Cloth. $5.00

Dziobek (O.). Die Grundlagen der Mechanik. Berlin, Mittler, 1907. 8vo. 4 + 345 pp. M. 6.00

Engel (P.). Etude géométrique du champ magnétique d'un navire. Paris, Chapelot, 1906. 8vo. 12 pp.

Felgentraeger (W.). Theorie, Konstruktion und Gebrauch der feineren Hebelwage. Leipzig, Teubner, 1907. 8vo. 6 + 310 pp. Cloth. M. 8.00

Fleming (J. A.). The principles of electric wave telegraphy. New York, Longmans, 1906. 8vo. 20 + 671 pp., 7 plates. Cloth. $6.60

Friocourt (G.). Tables de logarithmes à six décimales et tables de navigation. Nouvelle édition. Paris, 1907. 8vo. Fr. 10.00

Gennari (E.). Problema della temperatura stazionaria in una mezza sfera. Pesaro, Federici, 1906. 8vo. 38 pp., 2 plates.

Genovino (G.). Sulla convergenza dei meridiani. Genova, Gioventù, 1906. 8vo. 4 pp.

Gleijeses (M.). Equazione della superficie dei centri di galleggiamento per un solido qualunque. Lavoro originale. Napoli, Trani, 1906. 8vo. 27 pp. L. 1.00

——. Variazione del momento e del prodotto d'inerzia delle sezioni piane e parallele di un solido qualunque. Lavoro originale. Napoli, Trani, 1906. 8vo. 8 pp. L. 0.50

Grassi (G.). Principi scientifici della elettrotecnica. Introduzione al corso di elettrotecnica. Torino, Società Nazionale, 1907. 8vo. 6 + 320 pp. (Grande biblioteca tecnica, No. 3.) L. 12.00

Hiscox (G. D.). Modern steam engineering in theory and practice. London, Lockwood, 1907. 8vo. 488 pp. Cloth. 12s. 6d.

Hobbs (L. M.). Thermodynamic principles of engine design. London, Griffin, 1907. 12mo. 152 pp. Cloth. 4s. 6d.

Houstoun (R. A.). Untersuchungen über den Einfluss der Temperatur auf die Absorption des Lichtes in isotropen Körpern. (Diss.) Göttingen, 1906. 8vo. 53 pp., 2 plates.

Howe (M. A.). The design of simple roof-trusses in wood and steel, with an introduction to the elements of graphic statics. 2d edition, revised and enlarged. New York, Wiley, 1906. 8vo. 8 + 159 pp. Cloth. $2.00

Jeans (J. H.). An elementary treatise on theoretical mechanics. Boston, Ginn, 1907. 8vo. 8 + 364 pp. Cloth. $2.50

Krause (C.). Lehrbuch der Projectionslehre zum Schul- und Selbstunterricht. 3te Auflage. Berlin, Wendler, 1907. 8vo. 3 + 55 pp., 24 plates. M. 2.00

Láska (W.). Lehrbuch der Astronomie und der mathematischen Geographie. Teil I: Spharische Astronomie. Bremerhaven, 1907. 8vo. M. 5.00

Legrand. See Locht-Labye (de).

Lévy (M.). La statique graphique et ses applications aux constructions. 3e édition. Partie I: Principes et applications de statique graphique pure. Paris, Gauthier-Villars, 1907. 8vo. 30 + 598 pp., 26 plates. Fr. 22.00

Locht-Labye (de) et Legrand. Précis du cours de géométrie descriptive professé à l'Université de Liége. Partie I: Géométrie descriptive pure. 3e édition. 4to. 295 pp. Liége, Cluck, 1906. (Autographié.) Fr. 12.00

Lodge (O.). Electrons, or the nature and properties of negative electricity. London, Bell, 1907. 8vo. 246 pp. Cloth. 6s.

Meccanica applicata (r. Politecnico di Torino). Torino, Bertero, 1907. 8vo. 96 pp.

Mitinsky (A.). The elements of statics. Lectures. St. Petersburg, 1906. 8vo. (Russian.) M. 2.50

Model answers in matriculation mechanics together with the examination papers from June 1890 to June 1898 and from September 1902 to January 1907. London, Clive, 1907. 12mo. 228 pp. (University tutorial series.) 2 s.

Morasso (M.). Il nuovo aspetto meccanico del mondo. Milano, Hoepli, 1907. 16mo. 7 + 305 pp. L. 4.50

NICODEMI (R.). On the formula of the compound interest. Napoli, De Rubertis, 1906. 8vo. 6 pp.

REAN (F.). Ricerche sull'arco elastico. Torino, Gili, 1906. 8vo. 21 pp.

RICCI (O.). Cristallografia geometrica. Nozioni ad uso degli istituti tecnici e dei licei. Jesi, Flori, 1906. 8vo. 103 pp., 6 plates. L. 3.00

RIGHI (A.). La moderna teoria dei fenomeni fisici (radioattività, ioni, elettroni). 3a edizione, considerevolmente ampliata. Bologna, Zanichelli, 1907. 8vo. 11 + 290 pp. (Attualità scientifiche, No. 3.) L. 5.00

RÜDENBERG (R.). Energie der Wirbelströme in elektrischen Bremsen und Dynamomaschinen. Stuttgart, Enke, 1906. 8vo. 3 + 102 pp. (Sammlung elektrotechnischer Vorträge.) M. 3.60

RUSSELL (A.). Treatise on the theory of alternating currents. Vol. II. Cambridge, University Press, 1906. 8vo. 488 pp. Cloth. 12s.

SCHEFFLER (H.). Beiträge zur Kenntnis des Astigmatismus von Linsen. (Diss.) Halle, 1906. 8vo. 68 pp.

SCHULTZ (E.). Mathematische und technische Tabellen für Baugewerkschulen und für den Gebrauch in der Praxis. Ausgabe ohne Logarithmen. 7te Auflage. Essen, Baedeker, 1907. 8vo. 24 + 200 pp.

SCHWIDTAL. Technische Mechanik, nebst einem Abriss der Festigkeitslehre fur Bergschulen und andere technische Lehranstalten. 2te Auflage. Leipzig, Baedeker, 1906. 8vo. 7 + 87 pp. Boards. M. 1.50

SEIFFERT (O.). Vierstellige polygonometrische Tafeln zur Berechnung und Sicherung der Koordinatenunterschiede mit der Rechenmaschine. Braunschweig, Vieweg, 1907. 4to. 34 pp. Boards. M. 2.50

TAMMANN (G.). Ueber die Beziehungen zwischen den inneren Kraften und Eigenschaften der Lösungen. Ein Beitrag zur Theorie homogener Systeme. Hamburg, Voss, 1907. 8vo. 7 + 184 pp. M. 9.00

WÈVE (L.). Cinématique des mécanismes. Paris, 1907. 8vo. Cloth. Fr. 10.00

WINKELMANN (W.). Transformatoren und Asynchronmotoren. Ihre Wirkungsweise, Berechnung und Konstruktion. Hannover, 1907. 8vo. 10 + 136 pp. M. 4.20

THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and thirty-third meeting of the Society was held in New York City on Saturday, April 27, 1907. The April meeting is usually well attended, and ranks next in importance to the annual and the summer meetings. The present occasion was especially marked by Professor W. F. Osgood's presidential address, the subject of which was "The calculus in our colleges and technical schools." Under this attracting influence the attendance exceeded all previous records for the April meetings, reaching about seventy persons including the following sixty-one members of the Society:

Miss Grace Andrews, Professor G. A. Bliss, Professor Joseph Bowden, Professor E. W. Brown, Dr. W. H. Bussey, Dr. J. E. Clarke, Miss Emily Coddington, Professor F. N. Cole, Miss E. B. Cowley, Professor T. W. Edmondson, Professor L. P. Eisenhart, Professor F. C. Ferry, Professor T. S. Fiske, Dr. F. L. Griffin, Mr. G. W. Hartwell, Dr. G. W. Hill, Dr. A. M. Hiltebeitel, Dr. A. A. Himowich, Professor E. V. Huntington, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. G. H. Ling, Mr. Joseph Lipke, Mr. L. L. Locke, Dr. W. R. Longley, Professor E. O. Lovett, Mr. E. B. Lytle, Professor T. E. McKinney, Professor Max Mason, Dr. R. L. Moore, Professor Richard Morris, Professor W. F. Osgood, Dr. F. M. Pedersen, Miss Amy Rayson, Mr. H. W. Reddick, Dr. R. G. D. Richardson, Miss S. F. Richardson, Miss I. M. Schottenfels, Mr. L. P. Siceloff, Professor C. S. Slichter, Dr. Clara E. Smith, Professor D. E. Smith, Mr. F. H. Smith, Professor P. F. Smith, Dr. R. P. Stephens, Dr. C. E. Stromquist, Dr. W. M. Strong, Professor H. D. Thompson, Mr. M. O. Tripp, Professor H. W. Tyler, Dr. A. L. Underhill, Mr. C. B. Upton, Professor J. M. Van Vleck, Professor Oswald Veblen, Mr. H. E. Webb, Professor H. S. White, Miss E. C. Williams, Professor E. B. Wilson, Professor J. E. Wright, Professor J. W. Young.

The President of the Society, Professor H. S. White, occupied the chair. The Council announced the election of the following persons to membership in the Society: Dr. Alfred Ackermann-Teubner, Leipzig, Germany; Dr. J. W. Bradshaw, University of Michigan; Professor H. E. Cobb, Lewis

Institute, Chicago, Ill.; Mr. S. A. Corey, Hiteman, Ia.; Professor Floyd Field, Georgia School of Technology; Mr. G. W. Hartwell, Columbia University; Chancellor C. C. Jones, University of New Brunswick; Mr. Joseph Lipke, Columbia University; Professor Francis Regis, Christian Brothers College, St. Louis, Mo.; Mr. H. P. Stellwagen, Yeatman High School, St. Louis, Mo. Seven applications for admission to the Society were received. The total membership on May 1 was 560.

Following the plan announced in the Secretary's report of the February meeting, abstracts of the papers, as far as available, had been printed and issued with the programme in advance of the meeting. In this way it is hoped to secure a more intelligent interest in the papers and to promote criticism and discussion.

The date of the summer meeting, to be held at Cornell University, was fixed for Thursday–Friday, September 5–6.

The following papers were read at the April meeting:

(1) Professor G. A. Bliss: "A new form of the simplest problem of the calculus of variations."

(2) Professor R. D. Carmichael: "Multiply perfect even numbers of five different primes" (preliminary communication).

(3) Professor L. P. Eisenhart: "Transformations of surfaces whose lines of curvature are represented on the sphere by an isothermal system."

(4) Dr. F. L. Griffin: "The variation of the apsidal angle in certain families of central orbits."

(5) Dr. F. L. Griffin: "The solutions of central force problems as functions of the constant of areas."

(6) Dr. F. L. Griffin: "Note on a simple example of a central orbit with more than two apsidal distances."

(7) Professor G. A. Miller: "Note on the commutator of two operators."

(8) Professor J. E. Wright: "Arrangement of ovals of a plane sextic curve."

(9) Professor W. F. Osgood, Presidential address: "The calculus in our colleges and technical schools."

(10) Miss I. M. Schottenfels: "Group matrices."

(11) Dr. C. E. Stromquist: "An inverse problem of the calculus of variations."

(12) Dr. R. G. D. Richardson: "On the integration of a series term by term."

(13) Dr. A. L. Underhill: "Invariants of the function

$F(x, y, x', y')$ under point and parameter transformations connected with the calculus of variations."

(14) Professor EDWARD KASNER: "The motion of particles under conservative forces."

(15) Professor EDWARD KASNER: "Isogonal and dynamical trajectories."

(16) Professor P. L. SAUREL: "On the distance from a point to a surface."

(17) Professor T. E. MCKINNEY: "On concyclic quantics."

(18) Professor T. E. MCKINNEY: "On continued fractions representing quadratic irrationalities."

(19) Professor G. A. MILLER: "Groups generated by n operators each of which is the product of the $n - 1$ remaining ones."

In the absence of the authors the papers of Professor Carmichael, Professor Miller, Professor Saurel, and Professor McKinney were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. The simplest problem of the calculus of variations in the plane has been studied in two different forms. In the earlier one the curves considered are represented by equations $y = y(x)$, where $y(x)$ is a single-valued function, and the integral to be minimized is

$$I = \int f(x, y, y')\, dx. \tag{1}$$

In the later form, which seems first to have been studied in detail by Weierstrass, the curves are taken in parametric representation $x = \phi(t)$, $y = \psi(t)$, and the integral in the form

$$I = \int F(x, y, x', y')\, dt, \tag{2}$$

where the function F is positively homogeneous in the derivatives x' and y'. Both methods are open to objections, the former because it is inconvenient to apply to curves for which the function $y(x)$ is not single-valued or for which the slope becomes infinite, and the latter on account of the homogeneity condition $F(x, y, \kappa x', \kappa y') = \kappa F(x, y, x', y')$, $(\kappa > 0)$, which with its consequences must be kept in mind and frequently applied.

In a paper entitled "A generalization of the notion of angle" (*Transactions of the American Mathematical Society*, volume 7

(1906), page 184), Professor Bliss has introduced a third form of the problem in which the integral involves only invariants of the curve under change of parameter representation. The integral there given is

$$(3) \qquad I = \int f(x, y, \tau) ds,$$

where s is the length of arc along the curve, and τ is the angle which the tangent to the curve makes with the x-axis, defined by the equations

$$\cos \tau = \frac{x'}{\sqrt{x'^2 + y'^2}}, \quad \sin \tau = \frac{y'}{\sqrt{x'^2 + y'^2}}.$$

The integral (2) may be easily put into the form (3), for on account of the homogeneity of F,

$$F(x, y, x', y') = F(x, y, \cos \tau, \sin \tau)\sqrt{x'^2 + y'^2}.$$

But in (3) the function $f(x, y, \tau)$ need not be periodic as is the case in the right member of the last equation. Besides this generalization, which is important in applications to geometry such as Hamel's (see *Mathematische Annalen*, volume 57 (1903), page 231), the use of the integral in the form (3) avoids both of the objections made above to the other methods.

In the author's paper referred to above only the Euler equations and the equations defining transversality, which were necessary in finding the generalization of angle, were developed. In the present paper it is proposed to complete the theory of the problem in the form (3) by deriving the first and second variations and the functions which are commonly used in proving the usual necessary conditions in the calculus of variations (see Bolza, Lectures on the calculus of variations, pages 123, 133, 138; also Osgood, *Annals of Mathematics*, series 2, volume 2 (1901), pages 105 ff). The derivation of the conditions themselves can be made as is usual in the books on the subject.

2. By a rather difficult process, which he hopes to be able to simplify later, Professor Carmichael continues his investigations into the existence of multiply perfect numbers. In the present paper he takes up the case of seven numbers having only five different prime factors, finding that there are but four such numbers which are multiply perfect. One of these, $2^{13} \cdot 3 \cdot$

$11\cdot 43\cdot 127$, is of multiplicity 3; the others, $2^3\cdot 3^2\cdot 5\cdot 7\cdot 13$, $2^7\cdot 3^5\cdot 5^2\cdot 17\cdot 31$, $2^9\cdot 3^3\cdot 5\cdot 11\cdot 31$, are of multiplicity 4.

3. Ten years ago Thybaut* showed that minimal surfaces can be transformed into minimal surfaces in a manner analogous to the Bäcklund transformations of pseudospherical surfaces. A minimal surface and its Thybaut transform are the focal sheets of a *W*-congruence. The two minimal surfaces adjoint to these are the sheets of an envelope of spheres whose centers lie on a surface applicable to a paraboloid of revolution, by the theorem of Guichard. Darboux† and Bianchi‡ have considered the pairs of isothermic surfaces which are the envelope of a family of spheres and in conformal correspondence also, and the latter has given the name transformation of Darboux to a transformation of an isothermic surface which changes it into another such that the two form such a pair. Professor Eisenhart considers surfaces with isothermal representation of their lines of curvature and establishes a transformation of such a surface into another of the same kind. His transformation is a transformation of Darboux when the given surface is minimal and only in this case. Moreover, when the surface is minimal, this transformation is the same problem in analysis as the Thybaut transformation. By means of a theorem of Moutard it is shown that with each surface S whose lines of curvature have isothermal representation on the sphere, there are associated ∞^4 surfaces of the same kind, each of which forms with S the surface envelope of a family of spheres depending on two parameters. The determination of these new surfaces is the same problem apart from quadratures as the finding of the Thybaut transformations of the minimal surface with the same representation of its asymptotic lines as the lines of curvature of S. These transformations admit of a theorem of permutability similar to the theorems by this name shown by Bianchi to be true for Bäcklund and Darboux transformations. In consequence of this theorem, when all the transforms S_1 of a surface S are known, the transforms of the surfaces S_1 can be found without quadrature. Surfaces with plane lines of curvature are of the kind under discussion, as well as certain surfaces with spherical lines of curvature in one system. These

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* *Annales de L'Ecole Normale*, ser. 3, vol. 14, pp. 43–98.

† *Annales de L'Ecole Normale*, ser. 3, vol. 16, pp. 465–508.

‡ *Annali di Matem.*, ser. 3, vol. 11, pp. 93–157.

two classes of surfaces and their transformations are studied at length.

4. In central orbits the apsidal angle usually depends upon the apsidal values of the radius vector.* Considering a family of orbits, whose pericentral distance is constant and whose apocentral distance varies, Dr. Griffin discusses the definite integral giving the apsidal angle. He shows that for any force f which is a function of the distance r, such that fr^2 and its derivative with respect to r are both decreasing functions of r, the apsidal angle is a decreasing function of the apocentral distance.

An application is made to the motion of a particle in the equatorial plane of an oblate spheroid, explaining the more rapid "advance of the line of apsides" in the more nearly circular orbits.†

5. While the important part played by the constant of areas h in determining the nature of a central orbit has received some consideration, little attention has been given to the form in which h appears in the solutions. Dr. Griffin's second paper deals with a family of orbits with fixed pericenter, described under a central force, such that the solutions are expressible as power series. The coefficients are shown to be polynomials in $1/h$, having some interesting properties.

The results offer a working basis for the discussion of certain convergence questions. In this connection, as a check, the solutions for the newtonian law are examined also by the method devised by Professor Moulton ‡ for ascertaining the true radii of convergence for related series.

6. It is well known § that orbits described under a central force which is a single-valued function of the distance are symmetric with respect to every apsidal line where the radius vector is a maximum or minimum. Consequently the apsidal angle is constant and the orbit has not more than two apsidal distances. These conclusions, however, may not be drawn when the force is not a single-valued function of the distance.

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* Routh, E. J., Dynamics of a particle. §§ 368–370.

† Moulton, F.R., Celestial mechanics, pp. 232–233.

‡ *Astronomical Journal*, vol. 23, 1903, pp. 93–102.

§ Routh, E. J. Dynamics of a particle, pp. 270–272.

In this note, Dr. Griffin considers the ellipse, and shows that with certain interior points as centers of force, the orbit has three distinct apsidal distances and two distinct apsidal angles. The force has the peculiarity that it does not permit a real orbit in all parts of the plane.

7. Professor Miller's first paper aims to exhibit the different ways in which the term commutator has been used in the literature of the group theory, and to point out some of the sources of these disagreements in the hope that these data may tend towards greater uniformity in the use of this term and also make its various meanings less confusing to the reader. It is observed that the definition given by Weber in his Lehrbuch der Algebra, volume 2 (1899), page 133, differs from the earlier definition of Dedekind and Frobenius, and that some writers use a definition according to which the commutator is not completely determined by means of its elements. The conclusion reached is that it would be desirable to define $t^{-1}sts^{-1}$ and $s^{-1}tst^{-1}$ as the commutators of s, t and of t, s respectively.

8. Professor Wright shows that if a plane sextic have the maximum number (11) of ovals, they cannot be all external to one another, nor can they lie ten inside the eleventh. His proof consists first in modifying the equation of a sextic supposed to have the given arrangement of ovals so as to produce a sextic with ten isolated points. A further process of modification is next introduced whereby finally a sextic is obtained with eleven isolated points. Such a sextic would be reducible and it is easily seen that it cannot exist. Other arrangements of the ovals are also considered.

9. Professor Osgood's presidential address appears in full in the present number of the BULLETIN.

10. In the BULLETIN for November, 1906, pages 81–84, Dr. Lehmer exhibits a most orderly arrangement for the permutations on n letters. Two important laws are developed for the group matrices of the symmetric, alternating, and metacyclic groups upon n letters and these laws necessitate the development of but one-fourth of the matrix.

In the present paper Miss Schottenfels presents a further study of these particular group matrices, and also diagrams showing the symmetric positions of the matrices of the most important subgroups with respect to the original ones for $n = 2, 3, 4$, and 5.

11. The simplest problem in the calculus of variations is the following:[*] Given an integral

$$I = \int_{x_0}^{x_1} g(x, y, p)dx,$$

where $p = dy/dx$, to find a curve $y = y(x)$ which joins two given fixed points and renders this integral a minimum. As is well-known, the solutions $y = y(x)$ of the problem must satisfy Euler's[†] differential equation, which is a differential equation of the second order with a two-parameter family of solutions $y = y(x, \alpha, \beta)$ usually called extremals.

The so-called inverse problem arises when a two-parameter family of curves $y = y(x, \alpha, \beta)$ is given and we seek to determine the integral

$$I = \int_{x_0}^{x_1} g(x, y, p)dx$$

which shall have these curves for its extremals. This inverse problem has been discussed by Darboux,[‡] and it turns out that there are in general an infinite number of solutions.

In Dr. Stromquist's paper a third problem in connection with the integral I is considered, viz., if at every point of the plane a relation between the direction p of an extremal and the direction P transverse to it be given in the form (1) $P = \theta(x, y, p)$, is it then possible to determine an integral I whose extremals and transversals satisfy the given relation? It is shown that an infinitude of functions g can always be determined which will give rise to the relation (1). Further it can be required that a one-parameter family of curves $y = y(x, \gamma)$ be extremals for the integral I. Then it follows that g is uniquely determined provided that in the field composed of extremals $y = y(x, \gamma)$ the value of I as a function of x be given along any curve which is not tangent at any point to a transversal of the field.

In the second part of the paper the above results are applied to some examples.

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[*] Bolza, Lectures on the calculus of variations, § 3.

[†] Bolza, loc. cit., p. 22.

[‡] Darboux, Théorie des surfaces, vol. 3, p. 53. See also Bolza, loc. cit., p. 30.

12. Dr. Richardson's paper is in abstract as follows: Given a series $U(x) = u_1(x) + u_2(x) + \cdots$, what is a sufficient condition that integration term by term is permissible? Uniform convergence in a more or less modified form is the condition generally imposed, and the work of Osgood and Arzelà has this for a basis. For large classes of functions this condition is superfluous. Granted that the functions $U, u_1, u_2 \cdots$ are integrable, we may without any further condition write down three classes of series which are integrable term by term:

(*a*) The function $u_1, u_2 \cdots$ are all of the same sign ($u_i \geqq 0$, $i = 1, 2, 3, \cdots$; $u_i \leqq 0$).

(*b*) The series $\int |u_1| + \int |u_2| + \cdots$ converges.

(*c*) For all values of n the sum $\sum_1^n u_i$ is less in absolute value than some arbitrarily large but fixed number.

No restriction is put on the continuity of the functions $U, u_1, u_2 \cdots$, and in classes (*a*) and (*b*) these functions may take any values finite or infinite. An extension to functions of m variables is immediate.*

13. The object of Dr. Underhill's paper is the consideration of invariants, under point and parameter transformation, connected with the integrand function of the usual definite integral of the calculus of variations

$$I = \int_{t_0}^{t_1} F(x, y, x', y')dt.$$

The method developed is to start from a known invariant and from this, by a process not essentially different from the S-process of the calculus of variations, to derive new invariants.

Three invariants, denoted by S, K, V, are computed without restricting the arguments x, y, and their derivatives to those of an extremal. In case an extremal is used, a modified form of K is found and connected with it an invariantive normal form of the second variation of the integral I.

Finally S and K are interpreted for the problem of the geodesics on a surface. S is found to be the geodesic curvature; and K, when computed along an extremal, is the gaussian curvature. If the value of K is computed along a curve which is not an extremal, it becomes equal to the gaussian curvature minus one half the square of the geodesic curvature.

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* Osgood, "Non-uniform convergence and the integration of a series term by term" (*Amer. Journ. of Math.*, vol. 19.). Arzelà, Sulle serie di funzioni, Bologna, 1900.

Incidentally an expression for the Weierstrass function F_2 is found as well as a proof the Jacobi-Bonnet theorem : In the case of the problem of geodesics on a surface of negative curvature, the conjugate point is non-existent.

14. The theory of conservative forces worked out in Professor Kasner's first paper applies to any number of interacting particles, or to the equivalent problem of a single particle in a space of arbitrary dimensionality. The possible motions may be divided into families each characterized by a particular value of the constant of energy. To each value of this constant corresponds a definite system of trajectories, which is termed, following Painlevé, a natural system. The chief object of the paper is to find the geometric properties of such natural systems. The first result states that the circles of curvature constructed at a given point have another point in common. The point-to-point correspondence thus arising is not arbitrary ; its characterization is given in the second property. If a system has both properties, it is necessarily natural. The results apply to the related theories of geodesics (Darboux, Théorie des surfaces, volume 2, chapter 8) and of contact transformations (Lie, Berührungstransformationen, page 102, and *Leipziger Berichte*, 1889).

15. Professor Kasner's second paper deals with certain analogies between the theory of natural systems of dynamical trajectories in the plane, and systems of isogonal trajectories. In both cases the circles of curvature at a given point form a pencil. For isogonals the theorem is due to Cesàro and Scheffers. The types differ however with respect to the nature of the induced point transformation. The author obtains complete characterizations for both types ; connects the two theories by a certain transformation of curvature elements ; establishes a simple reciprocity between dynamical systems ; and finally proves that a natural system (and hence also a geodesic system) is completely determined by an arbitrary simple infinity of curves together with an arbitrary transversal curve.

16. Professor Saurel's paper appears in full in the present number of the BULLETIN.

17. In Professor McKinney's first paper are derived the sufficient conditions in terms of the invariants of the quantic that its roots, supposed distinct, be concyclic.

18. Professor McKinney's second paper discusses the properties of the continued fractions depending on a variable parameter λ, and representing quadratic irrationalities.

19. The principal theorems proved in Professor Miller's second paper are as follows: If the n different operators $s_1, s_2, \cdots, s_n$ are commutative and if each of them is the product of all the others, then they generate the direct product of a cyclic group whose order divides $2(n-2)$ and an abelian group of order 2^{α} and of type $(1, 1, 1, \cdots)$. Moreover, any such direct product may be generated by n operators which satisfy the given condition. If the order of one of these operators is divisible by 4, all of them are of the same order. In fact they always have a common square and the order of each divides $2(n-2)$. The necessary and sufficient condition that two different operators which have a common square are commutative is that one is the product of the other into an operator of order two.

If $s_1, s_2, \cdots, s_n$ are not supposed to be commutative but satisfy the condition $s_{\alpha}=s_{\alpha+1}s_{\alpha+2}\cdots s_n s_1 s_2\cdots s_{\alpha-1}$ $(\alpha=1, 2, \cdots, n)$ they may generate a large number of different types of groups when $n>4$. In particular, every possible symmetric group whose degree exceeds 3 may be generated by five such operators. All of these operators must have a common square and $s_{n-1}s_{n-2}\cdots s_2 s_1 = s_n s_1^{2(n-2)}$. The present paper determines the possible groups for the case when $n=4$, those for smaller values of n being known. It is proved that every group which may be obtained by extending the direct product of two cyclic groups by means of an operator of order two which transforms each operator of this direct product into its inverse may be generated by four operators of order 2, each of which is a product of the other three. A similar theorem is proved for the case when each of the operators s_1, s_2, s_3, s_4 is of order 4. It is observed that each of these four operators is always of even order and that the possible groups are somewhat similar to the dihedral type. In particular, all of them are solvable.

F. N. Cole,
Secretary.

THE MARCH MEETING OF THE CHICAGO SECTION.

THE twenty-first regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago, on Saturday, March 30, 1907. The total attendance was fifty, including the following thirty-eight members of the Society:

Professor W. W. Beman, Mr. G. D. Birkhoff, Professor O. Bolza, Mr. W. E. Brooke, Professor C. E. Comstock, Professor D. R. Curtiss, Professor E. W. Davis, Professor L. E. Dickson, Dr. E. L. Dodd, Professor L. W. Dowling, Mr. E. P. R. Duval, Mr. E. B. Escott, Dr. Peter Field, Professor A. G. Hall, Professor C. N. Haskins, Professor E. R. Hedrick, Mr. L. Ingold, Professor O. D. Kellogg, Dr. H. G. Keppel, Professor Kurt Laves, Mr. N. J. Lennes, Mr. W. D. MacMillan, Professor H. Maschke, Professor G. A. Miller, Professor E. H. Moore, Dr. J. C. Morehead, Professor F. R. Moulton, Dr. L. I. Neikirk, Mr. F. W. Owens, Professor D. A. Rothrock, Professor W. J. Rusk, Mr. A. R. Schweitzer, Professor E. B. Skinner, Professor H. E. Slaught, Professor E. J. Townsend, Professor E. B. Van Vleck, Dr. A. E. Young, Professor J. W. A. Young.

The chairman of the section, Professor E. B. Van Vleck, called the meeting to order and presented Professor H. Maschke, Vice-President of the Society, who presided at both the morning and afternoon sessions. It was voted that hereafter, at the discretion of the programme committee, two days be given to both the April and December meetings of the Section. A full meeting of the enlarged committee on programme for the December, 1907, meeting was held and definite plans were formulated for a symposium on the teaching of engineering mathematics.

The following papers were read:

(1) Dr. L. I. NEIKIRK: "A geometric representation of the Galois field."

(2) Mr. A. R. SCHWEITZER: "On the Ausdehnungslehre of Grassmann."

(3) PROFESSOR E. W. DAVIS: "A geometric construction for the imaginary intersections of a conic and a line" (preliminary communication).

(4) PROFESSOR L. E. DICKSON: "Representations of the general symmetric group as linear groups in finite and infinite fields."

(5) PROFESSOR E. B. VAN VLECK: "Note on curves without tangents."

(6) DR. A. E. YOUNG: "Asymptotic and isothermic surfaces."

(7) MR. LOUIS INGOLD: "Vector theory in terms of symbolic differential parameters."

(8) MR. E. B. ESCOTT: "New expressions for the number of classes of quadratic forms."

(9) PROFESSOR O. BOLZA: "Existence proof for a field of extremals tangent to a given curve."

(10) DR. PETER FIELD: "Note on a certain plane quintic curve."

(11) PROFESSOR L. E. DICKSON: "The symmetric group on eight letters and the senary first hypoabelian group."

(12) DR. C. H. SISAM: "On a property of parabolas and hyperbolas of mth degree."

(13) MR. LOUIS INGOLD: "Vector differential geometry in n-space."

(14) DR. E. L. DODD: "The iterated generalized limits of Cesàro and Borel."

(15) MR. G. D. BIRKHOFF: "Boundary value problems of ordinary linear differential equations."

(16) PROFESSOR G. A. MILLER: "Groups defined by the orders of two operators and the order of their commutator."

(17) PROFESSOR O. BOLZA: "Concerning conjugate points in the case of discontinuous solutions in the calculus of variations."

(18) PROFESSOR KURT LAVES: "Jacobi's partial differential equations, II."

(19) PROFESSOR R. D. CARMICHAEL: "On certain quartic curves which may degenerate into an ellipse."

(20) PROFESSOR R. D. CARMICHAEL: "A table of the numbers corresponding to given values of Euler's function $\phi(m)$."

(21) PROFESSOR R. D. CARMICHAEL: "On constructing a cube having a given ratio to a given cube."

(22) Professor C. C. GROVE: "The complete Pappus hexagon."

(23) PROFESSOR E. R. HEDRICK: "On implicit functions" (preliminary communication).

In the absence of the authors, the papers of Dr. Sisam and Professor Carmichael were read by title. Also, owing to the length of the programme, the second paper of Professor Bolza and the paper of Professor Hedrick were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles listed above.

1. In this paper Dr. Neikirk gives a construction of the algebraic number systems of the form

$$\omega = \sum_{r=1}^{r=n} C_r j^{n-r},$$

where j is a root of the irreducible equation $f(x) = 0$ and the C's are integral. This was accomplished on the ordinary form of complex plane. The results were then applied to finite fields, such as the Galois field, by distinguishing two kinds of points; *i. e.*, reduced points and congruent points.

2. Mr. Schweitzer observes that his n-dimensional geometric K_n system* ($n \geqq 3$) forms a convenient descriptive basis for the consideration of the fundamental principles of the Ausdehnungslehre. In fact, the statement $\alpha_1\alpha_2 \cdots \alpha_{n+1} K_n \beta_1\beta_2 \cdots \beta_{n+1}$ may be identified with the statement

$$s(\alpha_1\alpha_2 \cdots \alpha_{n+1}) = t(\beta_1\beta_2 \cdots \beta_{n+1}),$$

where s and t denote real numbers such that $st > 0$ and the indicated $(n+1)$-ads denote elementary magnitudes of the $(n+1)$st grade.

3. Professor Davis calls the point (x', y') a black point. It is joined by a red line to the point $(x' + x'', y' + y'')$ called a blue point. Thus x' and y' are the real parts of the coordinates, while x'' and y'' are the imaginary parts of the coordinates of the point $(x' + ix'', y' + iy'')$.

If there be a linear relation between $x' + ix''$ and $y' + iy'$ then there is in general a (1, 1) correspondence set up between black points and blue points in the plane. But if the blue line joining two blue points be parallel to the black line joining the corresponding black points, then all the points satisfying the linear relation and no others are satisfied by the elements repre-

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[*] Cf. BULLETIN, Nov., 1906.

sented by joining any point whatsoever of the black to any point whatsoever of the parallel blue line.

If the blue line coincide with the black line there results an ordinary real line.

4. The first paper by Professor Dickson gives a general investigation of the linear homogeneous groups on m variables with coefficients in a field F which are simply isomorphic with the symmetric group on q letters. In particular, it is shown that $m \geqq q-1$, unless F has a modulus dividing q. In the latter case there are representations on $q-2$ variables but not on fewer.

5. The paper of Professor Van Vleck, after briefly discussing a simple example of a continuous curve without a tangent, passed to the consideration of an example, simpler than any given hitherto, of a continuous curve which has an infinite number of oscillations in every part, however small, and yet possesses an ordinary tangent. This was followed by the following theorem: If in the interval (a, b) and in every subinterval the continuous curve $y = \phi(x)$ is of unlimited variation, there is a set of points of the second category, everywhere dense upon the curve, at which $\phi(x)$ has no derivative. This theorem may be considered as the complement of one previously obtained by Lebesgue for a continuous curve of limited variation.

6. Dr. Young called attention to the fact that certain of the known isothermic surfaces have isothermal asymptotic lines and undertook in his paper to show how all such surfaces may be obtained. He showed that the simplest treatment of the problem was from the standpoint of the spherical representation of the lines of curvature. For the problem reduces to that of finding all isothermic surfaces for which the spherical images of the lines of curvature are isotherms. Dr. Young obtained both the singular and general solution of an equation which expressed the condition of integrability of the Codazzi equations, and in this way reduced the problem to that of determining those forms of the arbitrary functions, thus obtained, which will satisfy the Gauss equation for the sphere. The singular solution leads to the quadrics, certain cyclides, certain surfaces associated with these, which Dr. Young has discussed in a previous paper, and finally to a number of new surfaces whose cartesian coordinates he has obtained. The latter furnish a clew

to two new sets of isothermic surfaces of which the surfaces found are particular cases. These he will report upon later.

The most general case arising from the general and singular solution combined Dr. Young has not completed, but he has carried the work far enough to indicate final success.

7. In his first paper Mr. Ingold exhibits a theory of vectors in a euclidean n-space S_n, as a special chapter of Professor Maschke's symbolic differental invariant theory. The derivatives $f_1, \cdots, f_n$ of the symbolic function f are identified with a set of n mutually orthogonal unit vectors of S_n; equivalent reference systems are introduced and symbolic expressions are obtained for λ-dimensional forms in S_n, inner and outer products, product determinant expansions, and regressive products. Finally the symbols of a λ-space A_λ contained in S_n are identified with a system of vectors tangent to the parameter curves of A_λ.

8. Lejeune Dirichlet has given expressions for the number of classes of forms of negative determinants $D = -P$ or $-2P$ in terms of the sum of the Jacobi symbols S/P for the octants and twelfths of P, and Dedekind has shown in his notes on Gauss's Works (Untersuchungen über höhere Arithmetik, pages 673–7, 693–5) how from these formulas Gauss's expressions for the distribution of the quadratic residues in the octants and twelfths of P may be obtained. In *Crelle's Journal*, volume 127 (1904), pages 1–19, Karpinski has extended the method of Dedekind to find the distribution of the quadratic residues in the 10ths and 24ths of P.

In the present paper, Mr. Escott has generalized the result of Dr. Karpinski for the division of P into $n$ths and has shown how these formulas will simplify in many cases the finding of the number of classes for a given determinant.

9. In his paper on "Sufficient condition for a minimum with respect to one-sided variations" (*Transactions*, volume 5, pp. 477–492) Professor Bliss has proved that under certain conditions a set of extremals tangent to a given curve form a field. This proof is indirect inasmuch as he proves the theorem first for the special case of curves representable in the form $y = f(x)$, and then reduces the general case of parameter representation to the former by a point transformation of the plane. The object of Professor Bolza's first note is to give a

direct proof of the theorem for the general case of parameter representation.

10. In a paper published in the last volume of the *Quarterly Journal of Mathematics* Mr. Bassett obtains the equation of a rational quintic curve having four cusps, from the fact that it is the dual of a rational quartic having one cusp. Dr. Field derived the equation and general form of the curve by inverting a tricuspid quartic which passes through the vertices and is tangent to one side of the triangle of reference.

11. Professor Dickson's second paper appeared in full in the May BULLETIN.

12. Dr. Sisam derived a formula for the number of tangents to an arbitrary algebraic curve such that the tangents at two intersections of any one of them with the curve intersect on the curve. He then showed that when the given curve is a parabola or hyperbola of mth degree, the tangents at any pair of points lying on the same tangent meet on the curve, and conversely. Several interesting properties of the correspondences thus established between the points and tangents of the curve were also discussed.

13. Mr. Ingold's second paper contains generalizations of the ordinary vector differential geometry as given by Burali-Forti and Fehr. A system of vectors tangent to the parameter lines is taken as reference system on an $(n-1)$-dimensional surface: these are determined from the cartesian representation by the methods of the first paper.

14. The iterated generalized limits of a multiple sequence may be unequal. Moreover, the existence of iterated generalized limits does not necessitate the existence of the generalized limit of a multiple sequence, nor vice versa. Dr. Dodd extends his treatment of iterated limits (*Mathematische Annalen*, volume 61, Heft 1) to the generalized limits of Cesàro and Borel.

15. The fundamental theorem of Mr. Birkhoff's paper is concerned with the character of the solutions of a differential equation

$$(1) \qquad \frac{d^n z}{dx^n} + \rho a_{n-1}(x, \rho) \frac{d^{n-1} z}{dx^{n-1}} + \cdots + \rho^n a_0(x, \rho) z = 0$$

for large values of the complex parameter ρ, $a \leqq x \leqq b$. Here for $|\rho| \geqq R$

$$a_i(x, \rho) = \sum_{j=0}^{\infty} a_{ij}(x) \cdot \frac{1}{\rho^j}.$$

The theorem holds on certain regions, $\theta \leqq \arg \rho \leqq \phi$, of the ρ-plane. With its aid is solved in large measure the problem to determine the distribution of those values of ρ for which a solution of (1) exists fulfilling n conditions

$$(2) \quad \sum_{j=0}^{n-1} \left\{ \alpha_{ij}(\rho) \frac{d^jz}{dx^j}(a, \rho) + \beta_{ij}(\rho) \frac{d^jz}{dx^j}(b, \rho) \right\} = 0 \quad (i = 1, 2, \cdots, n),$$

and to determine the character of these solutions. Here for $|\rho| \geqq R$

$$\alpha_{ij}(\rho) = \sum_{k=0}^{\infty} \alpha_{ijk} \cdot \frac{1}{\rho^k}; \quad \beta_{ij}(\rho) = \sum_{k=0}^{\infty} \beta_{ijk} \cdot \frac{1}{\rho^k}.$$

Using these facts, theorems analogous to those which Kneser has given in the case $n = 2$ are derived concerning the expansion of a function $f(x)$ in a series of solutions of a differential equation and conditions (like (1), (2))

$$(3) \quad \begin{gathered} p_n(x)\frac{d^nz}{dx^n} + p_{n-1}(x)\frac{d^{n-1}z}{dx^{n-1}} + \cdots + p_0(x)z + \rho^n A(x)z = 0, \\ \sum_{j=0}^{n-1} \left\{ \alpha_{ij} \frac{d^jz}{dx^j}(a, \rho) + \beta_{ij} \frac{d^jz}{dx^j}(b, \rho) \right\} = 0 \qquad (i = 1, 2, \cdots, n) \end{gathered}$$

in the real self- or opposite-adjoint case, $p_n(x) \neq 0$, $A(x) \neq 0$. It is also proved that this restriction to the real self- or opposite-adjoint case are to a great extent unessential.

16. The object of Professor Miller's paper is to develop theorems which will make the commutator a more useful instrument in the study of group properties. The results are analogous to those which relate to the groups of genus zero. The following theorems are proved: Any dihedral group whose order is divisible by 8 may be defined by the orders of two generators and the order of the commutator of these generators, and no other groups are known to have this property. There is an infinite system of groups such that each of them is generated by two operators whose commutator is of order 2 while the order of one of the generators is any given number which

exceeds 3 and the order of the other is any given even number greater than zero. If the orders of the two generators of a group are either 3, 4 or 3, 5 and if the order of their commutator is 2, these generators may be so selected as to give rise to an arbitrary group in an infinite system. If a group is generated by two operators of orders 2 and 3 respectively whose commutator is of order 3, its commutator subgroup is generated by two operators of order 3 whose product is also of this order. There are three and only three such groups which have an abelian commutator subgroup. The orders of these groups are 6, 18, and 54 respectively.

17. Professor Bolza's second note supplements Dr. Carathéodory's investigation concerning conjugate points for discontinuous solutions in the calculus of variations, by giving in explicit form the equation which connects the parameters of a pair of such conjugate points.

18. In a previous paper by Professor Laves on the same subject (presented at the Chicago meeting of April, 1906) it was shown that the partial differential equation for a body rotating about a fixed point can be integrated by determining the action of the system between two terminal positions. It is here shown how this proceeding rests on a theorem of Jacobi for problems depending upon two parameters. A family of orthogonal curves, one set of which are the extremals for the same value of the constant of energy, while the other are the transversals, covers the plane. The action of the system between two positions A_0 and A_1 is the difference between the values of the transversals passing through these same points. This is the form given to Jacobi's theorem by Sir William Thomson.

19. Professor Carmichael's first paper appeared in full in the March number of the *American Mathematical Monthly*.

20. In this table Professor Carmichael gives every possible value of Euler's function $\phi(m)$ up to 1000, and lists opposite each value all the numbers m which correspond to it. It is now more than 30 years since Cayley expressed a desire for the construction of this table. It will, it is believed, be found useful in various ways in the study of number theory.

21. In his third paper Professor Carmichael constructs the semicubical parabola by continuous motion, and applies the

locus to finding a cube having to a given cube any ratio which may be expressed by two straight lines.

22. The proposition, "If A, B, C are three points of one line and A', B', C' are three points of another line, then the intersections, $BC'/B'C$, $CA'/C'A$, $AB'/A'B$ lie on a line" (Salmon: Conics, § 268) puts in simple form Propositions 138, 139, Liber VII, Collectiones of Pappus. The figure arising is a special case of the Pascal hexagon which may be called the Pappus hexagon. Professor Grove's paper develops the complete Pappus hexagon and shows the duality of the figure in the following and other theorems with their duals:

I. Three points π_i on each of two lines D_1, D_2 meeting in the point δ_1 are cross-joined by nine lines which meet in 18 points 1–18, which are the vertices of two sets of three line triads each. The triangles of each set are *inter se* in triple perspective, having as axes of perspective the lines D_1 and D_2 each three times and three lines P_i on the point Σ_2 or Σ_1 respectively, and having as centers of perspective the point Σ_1 or Σ_2 three times each for the sets respectively and twelve other points all of which lie on a covariant line E_1 which is the polar of δ as to any of the six triangles. The 18 points lie by threes on six lines P_i, which are three and three on the points Σ_1 and Σ_2 above.

II. The points Σ_1 and Σ_2 and δ are the false vertices of the complete quadrangle of the Hessian pairs of the point triads π_i on D_1 and on D_2.

Following names given in the complete Pascal hexagon, he calls the three points π_1, π_3, π_5 on D_1, and π_2, π_4, π_6 on D_2 Pappus vertices; the three lines P_i on Σ_1 and Σ_2 each, he calls Pappus lines; the points Σ_1 Steiner points whose line is S_i, the lines D_1 Hessian diagonals. Taking the lines P_i from the above theorem and proceeding in the dual form, we get three points π'_i on each of the *same* two lines D_i etc., etc. The derived triads π'_i and the counter-triads of the original triads are in an involution whose double points are the polar pair of the intersection δ as to the original triads π_i. Six other theorems follow which deal with special forms.

This paper will appear in full in the May number of the *American Mathematical Monthly.*

23. In this paper Professor Hedrick shows that the ordinary statements of the fundamental theorem on the existence of a solution of the implicit equation $F(x, y) = 0$ and their demon-

strations are included in the statement and demonstration of the more general and simple theorem :

If $F(x, y) = 0$ is satisfied by $x = x_0$, $y = y_0$, and if $F(x, y)$ is a continuous function of x and a continuously increasing (or decreasing) function of y near (x_0, y_0), then there exists one single-valued solution $y = \phi(x)$ near (x_0, y_0) such that $y_0 = \phi(x_0)$ and $F(x, \phi(y)) = 0$; and that solution is continuous.

A discussion of the relation of this theorem to other forms was appended.

H. E. SLAUGHT,
Secretary of the Chicago Section.

ON A LIMIT OF THE ROOTS OF AN EQUATION THAT IS INDEPENDENT OF ALL BUT TWO OF THE COEFFICIENTS.

BY PROFESSOR R. E. ALLARDICE.

(Read before the San Francisco Section of the American Mathematical Society, February 23, 1907.)

AT the end of a paper by Dr. Landau,[*] it is shown that every equation of the form $ax^n + x + 1 = 0$ has a root whose modulus is not greater than 2, and that every equation of the form $ax^n + bx^m + x + 1 = 0$ has a root whose modulus is not greater than 8. The object of the present paper is to show that every equation of the form

$$ax^n + bx^m + cx^l + \cdots + a_1x + a_0 = 0$$

has a root whose modulus is not greater than

$$\left|\frac{a_0}{a_1}\right| \cdot \frac{n}{n-1} \cdot \frac{m}{m-1} \cdot \frac{l}{l-1} \cdots$$

whatever be the values of the coefficients $a, b, c, \cdots$; and that, for certain values of these coefficients, this limit is attained.

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[*] "Ueber den Picardschen Satz," *Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich*, Jahrgang 51, 1906.

It is obvious that by reason of the substitution $a_1x = a_0y$, we may take $a_1 = a_0 = 1$.

The method of proof lies in showing that, by taking appropriate increments of the arbitrary coefficients, we may increase the modulus of any root of the proposed equation, unless the root in question is one of a set of equal roots, the number of which is greater by one than the number of the arbitrary coefficients. The sole difficulty lies in the consideration of roots of equal modulus, but different amplitudes.

1) Consider the equation

$$ax^n + x + 1 = 0.$$

Let $\rho\alpha$, $\rho\beta$ be two roots with common modulus ρ $(\alpha \neq \beta)$; then

$$a\rho^n\alpha^n + \rho\alpha + 1 = 0, \quad a\rho^n\beta^n + \rho\beta + 1 = 0.$$

$$\therefore \rho(\alpha^n\beta - \alpha\beta^n) + \alpha^n - \beta^n = 0$$

and

$$\rho(\alpha^{-n}\beta^{-1} - \alpha^{-1}\beta^{-n}) + \alpha^{-n} - \beta^{-n} = 0,$$

whence

$$\rho(\alpha\beta - 1)(\alpha^{n-1} - \beta^{n-1}) = 0.$$

If $\alpha^{n-1} - \beta^{n-1} = 0$, it follows also that $\alpha^n - \beta^n = 0$, which is impossible; hence $\alpha\beta - 1 = 0$, or the two roots are conjugate, and a must be real.

Now, putting $x = \rho(\cos\theta + i\sin\theta)$ in the given equation, equating to zero the real and imaginary parts, taking differentials, and eliminating $d\theta$, we may easily show that it is possible to increase the modulus of each of the roots $\rho e^{\theta i}$, $\rho e^{-\theta i}$ by giving a real increment to a. The coefficient of $d\rho$ in the relation between $d\rho$ and da cannot vanish.

If, however, the proposed equation have two equal roots, an increment of a that will increase the modulus of one of these roots will diminish that of the other. Hence the proposed equation has always a root whose modulus is not greater than $n/(n-1)$, which is the value of the equal roots.

2) Consider now the equation

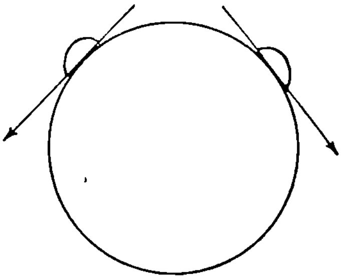
$$ax^n + bx^m + x + 1 = 0.$$

Let x_1, x_2 be two roots of equal modulus, neither of which is a multiple root; then

$$dx_1 = -\frac{x_1^n}{nax_1^{n-1}+mbx_1^{m-1}+1}\,da - \frac{x_1^m}{nax_1^{n-1}+mbx_1^{m-1}+1}\,db$$

$$= p_1 da + q_1 db, \quad \text{(say)}$$

$$dx_2 = p_2 da + q_2 db.$$



Now, representing x_1 and x_2 on a circle with radius equal to the common modulus, we see that $|x_1|$ and $|x_2|$ are increased if dx_1 and dx_2 lie in the spaces indicated in the figure.

We have

$$dx_1 = da\left(p_1 + q_1\frac{db}{da}\right), \quad dx_2 = da\left(p_2 + q_2\frac{db}{da}\right),$$

and hence, if db/da be taken arbitrarily, dx_1 and dx_2 may be made to rotate through four right angles, by varying the amplitude of da. It follows that, for some value of da, $|x_1|$ and $|x_2|$ will both be increased unless when dx_1 coincides with the tangent at x_1, dx_2 also coincides with the tangent at x_2 (in the directions indicated in the figure).

The conditions for these coincidences are

$$p_1 + q_1\frac{db}{da} = k_1 x_1 e^{-\frac{\pi}{2}i} e^{\alpha i}, \quad p_2 + q_2\frac{db}{da} = k_2 x_2 e^{\frac{\pi}{2}i} e^{\alpha i},$$

where k_1 and k_2 are real and have the same sign.

$$\therefore \frac{p_1 da + q_1 db}{p_2 da + q_2 db} = -\lambda\frac{x_1}{x_2} \quad (\lambda \text{ real and positive}).$$

Hence the fraction on the left must be independent of da and db, otherwise it would be possible to make it equal to any complex number.

$$\therefore p_1/q_1 = p_2/q_2, \quad \text{whence} \quad x_1^{n-m} = x_2^{n-m};$$

$$\therefore x_2 = \epsilon x_1, \quad \text{where} \quad \epsilon^{n-m} = 1.$$

It is easy now to deduce the values

$$x_1 = (1 - \epsilon^m)/(\epsilon^n - \epsilon), \quad x_2 = (1 - \epsilon^m)/(\epsilon^{n-1} - 1),$$

from which it follows that x_1 and x_2 are conjugate.

Denoting the conjugates of a and b by $\bar{a}$ and $\bar{b}$, we see that x_1 and x_2 are roots of the equations

$$ax^n + bx^m + x + 1 = 0, \quad \bar{a}x^n + \bar{b}x^m + x + 1 = 0$$

and hence of

$$(a\bar{b} - \bar{a}b)x^m + (a - \bar{a})x + (a - \bar{a}) = 0\,;$$

and the moduli of x_1 and x_2 may be increased together by the results obtained for the equation first considered. If a and b are both real, it may be shown as before, by putting $x = \rho(\cos\theta + i\sin\theta)$, differentiating and eliminating $d\theta$, that the moduli of x_1 and x_2 may both be increased by taking real increments of a and b.

Hence the modulus of any root of the given equation may be increased, unless it is a double root. But any double root of the given equation is a root of the equation

$$(n - m)bx^m + (n - 1)x + n = 0,$$

and may therefore have its modulus increased, unless it be a double root of this latter equation and therefore a triple root of the original equation. Thus the proposed equation always has a root whose modulus is not greater than $nm/(n - 1)(m - 1)$.

3) Consider now the equation

$$ax^n + bx^m + cx^l + x + 1 = 0.$$

As before, any two unequal roots of equal modulus may have their modulus increased unless

$$p_1 : q_1 : r_1 = p_2 : q_2 : r_2.$$

This leads to the equations

$$x_1^{n-m} = x_2^{n-m}, \quad x_1^{m-l} = x_2^{m-l},$$

which are impossible unless $n - m$ and $m - l$ have a common factor.

Let

$$n - m = k_1 r, \quad m - l = k_2 r.$$

$$\therefore n = (k_1 + k_2)r + l, \quad m = k_2 r + l$$

and

$$x_2 = \epsilon x_1 \quad \text{where} \quad \epsilon^r = 1.$$

We may easily show that x_1 is determined by the equation

$$(\epsilon^l - \epsilon)x_1 + \epsilon^l - 1 = 0,$$

and that x_1 and x_2 must be conjugate; and the investigation may be completed as in the last case.

It is obvious that the above method may be continued so as to include equations containing any number of terms.

It may be stated in conclusion that the problem solved in the present paper is connected with the more difficult problem of determining a quantity ρ, a function of a_0 and a_1, such that there shall always be a root either of the equation $f(x) = a$ or of the equation $f(x) = b$, with modulus less than ρ, and that this latter problem is connected with the theorem of Picard, which is discussed in Dr. Landau's paper.

STANFORD UNIVERSITY,
February, 1907.

ON THE DISTANCE FROM A POINT TO A SURFACE.

BY PROFESSOR PAUL SAUREL.

(Read before the American Mathematical Society, April 27, 1907.)

IT is well known that in order that the distance from a given point to a given surface be a maximum or a minimum it is necessary that this distance be measured on a normal to the surface. But, so far as I know, the various possible cases have not been enumerated. This is done in the following theorem:

If P be an elliptic point of a surface, and if C_1 be the nearer and C_2 the more remote of the principal centers of curvature, the distance from a given point N of the normal to P will be a minimum if N and P lie on the same side of C_1, a maximum if N and P lie on opposite sides of C_2, and neither a minimum nor a maximum if N coincide with C_1 or C_2, or lie between them.

If P be a hyperbolic point of a surface, and if C_1 and C_2 be the principal centers of curvature, the distance from a given point N of the normal to P will be a minimum if N lie between C_1 and C_2, and neither a minimum nor a maximum if N coincide with C_1 or C_2, or lie without the interval C_1C_2.

If P be a parabolic point of a surface, and if C_1 be the principal center of curvature, the distance from a given point N of the normal to P will be a minimum if N and P lie on the same side of C_1, and neither a minimum nor a maximum if N coincide with C_1 or if N and P lie on opposite sides of C_1.

The demonstration rests upon the following theorem in plane geometry: In order that the distance from a given point to a given curve be a maximum or a minimum it is necessary that this distance be measured on a normal to the curve. The distance is a minimum if the given point N and the foot of the normal P lie on the same side of the center of curvature C, a maximum if they lie on opposite sides, and neither a minimum nor a maximum if N coincide with C.

We shall assume that this theorem is known. It may be well, however, to point out that it becomes almost intuitive if we observe that, in the immediate neighborhood of P, the curve lies outside of any tangent circle whose center lies on the same side of C as P, that it lies inside of any tangent circle whose center lies on the opposite side of C, and that it crosses the tangent circle whose center is at C.

The theorem in solid geometry follows at once from this theorem in plane geometry if we remember that for an elliptic point the centers of curvature of the normal sections lie within the interval C_1C_2, that for a hyperbolic point they lie without this interval, and that for a parabolic point they lie on that part of the normal which does not cut the surface.

It is worth noticing that the third part of the theorem can be considered as a limiting form of either the first or the second part.

NEW YORK,
April 13, 1907.

THE CALCULUS IN OUR COLLEGES AND TECHNICAL SCHOOLS.

PRESIDENTIAL ADDRESS DELIVERED BEFORE THE AMERICAN MATHEMATICAL SOCIETY APRIL 27, 1907.

BY PROFESSOR WM. F. OSGOOD.

THE history of the race is frequently suggestive, in educational matters, of a wise course for the training of the individual. If we turn to the problems to which the calculus owes its origin, we find that not merely, not even primarily, geometry, but every other branch of mathematical physics — astronomy, mechanics, hydrodynamics, elasticity, gravitation, and later electricity and magnetism — in its fundamental concepts and basal laws contributed to its development and that the new science became the direct product of these influences.

Not until recent times have the analytic methods of the calculus been securely established. But at last the method of rates and that of infinitesimal constants have given place to the method of limits as being the only method known to us on which the calculus can be satisfactorily founded, satisfactorily, not merely from a logical point of view; it is important that the methods be in close touch with the physical concepts with which the calculus deals.

What is the bearing of these facts on instruction in the calculus in our colleges and technical schools? I wish to deal with some of the phases of this question.

§ 1. *Object of the Study of the Calculus.*

Let us first make clear to ourselves what some of the chief aims of the study of the calculus are. A view that has been widely accepted, to judge from the text-books that have had the largest sale, is that the first course should deal almost solely with the formal side. Just as the school boy learns to manipulate algebraic expressions with skill and accuracy, so the student shall first be drilled in differentiation and integration, the applications being chiefly to geometry and consisting in computing by formulas deduced in the text-book or class-room.

Recently the pendulum has swung to the other extreme. The reformists have discovered that the engineer is well off if he can plot some simple curves from the tables and differentiate and integrate x^2 and $\sin x$, and we are told that the race has been degenerating under the old regime. Let us not lose perspective and let us not for a moment fail to recognize the fact that, whatever changes it may be desirable to make in the suggestive instruction of the course, the process by which the youth actually acquires the ideas of the calculus is to a large extent and essentially through formal work of substantial character. In order to attain this end, however, the formal work must appear to him as having for its direct object the power to solve some of the real problems of pure and applied mathematics, and these problems must always be kept before his eye.

Setting aside such secondary aims as the ability to recognize the derivatives and integrals that enter in the text-books of applied mathematics, let us ask ourselves what enduring service the study of the calculus can render to the man whose life work calls for a qualitative, and, perhaps, to some extent quantitative appreciation of physical phenomena. I believe one of the most valuable contributions of the calculus toward this end to lie in the training by which the student ultimately reaches the point of being able to discern and formulate those concepts and relations of physics that find their natural expression in definite integrals and differential equations. The road to the qualitative is through the quantitative. To be conscious that the flight of a golf ball is governed by Newton's laws of motion and to discern what the forces are that act and how they bring about the slice, is to have taken an important step toward dealing intelligently with dynamical questions in the construction of machinery or of a steamship. But the problem of the golf ball should not come till many, many simple quantitative problems in rectilinear and circular motion have been worked out by the student.

§ 2. *Differentials, Ancient and Modern.*

The original conception of the infinitesimal was that it is a very small, an "infinitely small," *constant* quantity, but not quite the absolute zero, a sort of *little zero* that was insignificant when compared with any of the quantities of every day life except zero, but that two of these differentials could be compared with each other in their microcosmos just as the quantities of every day life are comparable among themselves. Now this conception had its origin in a partial discernment of the truth, but, like the asymptotic series of astronomy, which converge only to a certain point and then diverge, it must not be pressed to the limit; for then the truth ceases and only chaos remains. Thus mathematicians have necessarily discarded the differentials of Leibniz as the elements out of which the calculus can be built up, and some are more than doubtful about the advisability of retaining them in any form.

We sometimes hear it said that hardly a theorem in our textbooks on the calculus is true as stated there. This is only a partial truth. Most of the theorems in the calculus are true when restricted as the authors intended they should be or as modern views readily suggest. And so here, in the case of the differentials, it behooves us to extract that which is true in the crude conception of former times and to shape a modern conception that is rigorous according to our present light.

There is a cogent reason for this course. The differentials are as firmly seated as ever they were in the methods of the physicists and engineers. We cannot banish them there, even if it were desirable to do so. Leave them out of the course in calculus and the student will meet them there with all the contradictions of the archaic infinitesimals, the little zeros of former times; and if he is capable of thinking clearly, he will see these contradictions and be helpless to reconcile them.

But beside this material reason there are two good scientific reasons for keeping the differentials and the infinitesimals. The former are useful in the formal work of differentiation and integration, the latter in the formulation of physical problems.

§ 3. *The Modern Conception of Infinitesimals.*

We have considered the archaic view of infinitesimals as small constants — a view still represented in the language, as when, for example, the President says: "The United States, owing to its peculiar position, has a regular army so small as to be *infinitesimal* when compared to that of any other first-class power." The modern scientific conception is that an infinitesimal is a *variable* which it is generally useful to consider only for values numerically small and which, when the formulation of the problem has progressed to a certain stage, is then allowed to approach zero as its limit.

Two infinitesimals, α and β, are said to be of the same *order* if their ratio, β/α, approaches a finite limit $K \neq 0$. If $K = 0$, β is said to be of higher order than α; and if $K = \infty$, then β is of lower order. If

$$\lim \frac{\beta}{\alpha} = 1, \quad \text{then} \quad \frac{\beta}{\alpha} = 1 + \zeta,$$

where ζ is infinitesimal, and hence α differs from β by an infinitesimal $\zeta\alpha$ of higher order than α or β. Conversely, if α or β differ from each other by an infinitesimal of higher order than that of either, then the limit of their ratio is unity. That infinitesimal which is chosen as independent variable is called the *principal infinitesimal.* If this be α, then, since

$$\lim \frac{\beta}{\alpha} = K, \quad \frac{\beta}{\alpha} = K + \zeta,$$

and

$$\beta = K\alpha + \zeta\alpha.$$

Thus β differs from $K\alpha$ by an infinitesimal of higher order than α; $K\alpha$ is called the *principal part* of β.

Two theorems lie at the foundation of the application of the calculus to natural philosophy.

THEOREM 1. In the limit of the ratio of two infinitesimals, either infinitesimal may be replaced by one that differs from it by an infinitesimal of higher order:

$$\lim \frac{\beta}{\alpha} = \lim \frac{\beta'}{\alpha'} \quad \text{if} \quad \lim \frac{\beta'}{\beta} = 1, \quad \lim \frac{\alpha'}{\alpha} = 1.$$

THEOREM 2. *Duhamel's Theorem.* If a sum of infinitesimals, all of like sign, approaches a limit, then each of the infinitesimals may be replaced by one that differs from it by an infinitesimal of higher order:

$$\lim_{n=\infty} (\alpha_1 + \alpha_2 + \cdots + \alpha_n) = \lim_{n=\infty} (\beta_1 + \beta_2 + \cdots + \beta_n),$$

if

$$\lim_{n=\infty} \frac{\beta_i}{\alpha_i} = 1.$$

§ 4. *The Use of Infinitesimals in Formulating Physical Problems.*

The importance of infinitesimals in the course in calculus which we are here considering lies in their use in formulating physical problems. A typical example or two will serve to make my meaning clear.

Let it be required to find the tension in a rope wound round a rough post and just on the point of slipping.

Consider a small arc PP' of the rope. We may regard it as a rigid body, since, if it became stiff, the forces would not necessarily thereby be disturbed. The forces that act on this piece of the rope are: (1) the tension T along the tangent at P; (2) the tension $T+\Delta T$ along the tangent at P'; and (3) a resultant reaction S of the cylinder; — the weight of the rope is unimportant in comparison with the forces considered. Resolving forces along the tangent and the normal at P, we get *

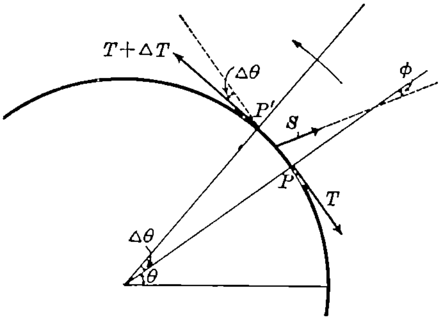


FIG. 1.

$$\left.\begin{aligned}(T+\Delta T)\cos\Delta\theta &= S\sin\phi + T,\\ (T+\Delta T)\sin\Delta\theta &= S\cos\phi.\end{aligned}\right\}\quad(1)$$

Now if the rope is just on the point of slipping in the direction of the increasing angle θ, friction will act in the direction of the decreasing angle, and thus S will be inclined as shown in Fig. 1. What hypothesis shall we make about the direction of S? In other words, what is the physical fact? It is easy to see if we think of a heavy flexible string AB laid out straight on a rough horizontal table and pulled at one end A

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* If the class is not familiar with the fact that, when a system of forces, acting on a rigid body, keep it at rest, they must be such a system as would be in equilibrium if they all acted at a single point, — then this theorem must, of course, be stated.

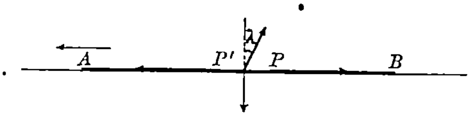

FIG. 2.

till it is just on the point of slipping. The total reaction S of the table on a segment PP' here obviously makes an angle ϕ with the normal equal to the angle of friction λ.

In the case of the cylinder we have similar relations. The rope presses against the cylinder along the arc PP', which, being short, is but slightly curved, and hence the direction of S will make an angle ϕ with the normal at P which differs only slightly from the angle of friction λ, and approaches λ as its limit when P' approaches P.

From equations (1) it follows that

$$\tan\phi = \frac{(T+\Delta T)\cos\Delta\theta - T}{(T+\Delta T)\sin\Delta\theta}$$

$$= \frac{\cos\Delta\theta}{T+\Delta T}\,\frac{\Delta T}{\sin\Delta\theta} - \frac{T}{T+\Delta T}\,\frac{1-\cos\Delta\theta}{\sin\Delta\theta}.$$

We are now ready to allow P' to approach P as its limit:

$$\lim_{PP'=0}\tan\phi =$$

$$\left(\lim_{PP'=0}\frac{\cos\Delta\theta}{T+\Delta T}\right)\left(\lim_{PP'=0}\frac{\Delta T}{\sin\Delta\theta}\right)-\left(\lim_{PP'=0}\frac{T}{T+\Delta T}\right)\left(\lim_{PP'=0}\frac{1-\cos\Delta\theta}{\sin\Delta\theta}\right).$$

The limit of $\tan\phi$ is μ, the coefficient of friction. The limit of the first factor in each product is obvious, namely $1/T$ and 1 respectively; for the tension increases gradually as we go along the rope, and so ΔT approaches 0 when P' approaches P.* In the limit of the second factor of the first term on the right, $\sin \Delta\theta$ can, by Theorem 1 of § 3, be replaced by $\Delta\theta$ and hence

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* From equations (1) it can be proven that ΔT approaches 0 simultaneously with $\Delta\theta$; but the important view for the student at this stage is that the function T changes continuously because it represents a physical quantity that, from its very nature, varies continuously.

$$\lim_{\Delta\theta=0} \frac{\Delta T}{\sin \Delta\theta} = \lim_{\Delta\theta=0} \frac{\Delta T}{\Delta\theta} = D_\theta T.$$

The same replacement can be made in the last limit on the right; and since

$$\lim_{\alpha=0} \frac{1 - \cos \alpha}{\alpha} = 0,$$

the limit of the whole last term on the right is 0. Our final result is, therefore,

$$\mu = \frac{D_\theta T}{T},$$

or

$$\frac{dT}{T} = \mu d\theta.$$

Hence, integrating, we get

$$\log T = \mu\theta + C,$$

and thus finally

$$T = T_0 e^{\mu\theta}.$$

In the foregoing solution, which I have given in all detail, it will be observed that, in the formulation of the problem, the physical law is clearly stated: ϕ is approximately equal to λ when P' is near P, and approaches λ as its limit when P' approaches P. A difficulty which the student frequently meets in this border ground between mathematics and physics is that of finding out what the physical fact is. The physicist often shrinks from writing down a formula for fear that the student will not understand what it means, quite losing sight of the fact that it is just at this point that the student can most effectively be aided in forming clean cut physical conceptions, the natural expression of which is through mathematical symbols.

The physical law once granted, equations (1) are absolutely true, whether the arc PP' subtends an angle of 10° or .000001″. Moreover, in taking the limits, nothing is "neglected." And the limits lead us directly to *derivatives, not to differentials*. The differentials enter only subsequently through the derivatives. If the differentials are introduced at the outset, θ being taken as the independent variable, the force at P', namely $T + \Delta T$, is not $T + dT$; it differs from the latter quantity by an infinitesimal of higher order. And the equations (1), written with the Δ's replaced by the d's and ϕ by λ, are not true till a correction in the form of an infinitesimal of higher order has been added to one side of each equation. Thus the use of differentials in formulating the problem, if carried out rigorously, only detracts from the simplicity of the equations and at best replaces the quantities that naturally enter by artificial ones not quite equal to them. In the formulation of physical problems leading to definite integrals—areas and volumes, centres of gravity, moments of inertia, fluid pressures, attraction of gravitation, work and energy, kinetic energy, etc.—the situation is similar. First, the quantity to be computed is divided into n *infinitesimal* parts, *i. e.*, into n definite pieces, the magnitude of each of which is expressed accurately by an analytic symbol, and the quantity in question is precisely the sum of these pieces. Hence it is also equal to the limit of their sum. Then the ith part is approximated to by means of an infinitesimal which, in the case of a simple integral, is of the form $f(x_i)\Delta x_i$ and which differs from the ith infinitesimal in the above sum by an infinitesimal of higher order. By virtue of Duhamel's theorem the limit of the second sum,

$$\lim_{n=\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i,$$

is equal to that of the first sum, and thus the quantity to be computed is formulated as the definite integral

$$\int_a^b f(x)dx.$$

Here, again, nothing is "neglected." There is a clean cut formulation of the quantity in question as a sum, hence as the limit of that sum, and then follows the evaluation of this limit by means of a definite integral.

These are two of the central problems of the calculus, and an intelligent solution of them depends on a clear and adequate understanding of what infinitesimals are and how to use them. No greater mistake can be made than to think that these problems belong in the applications of the calculus in the courses in physics and engineering. This view is a wrong one for two reasons. First, these problems are essential for understanding what the calculus is. Poincaré has said that it is nature which forced the continuum on mathematicians. If they had been left the prey of abstract logic, they would never have gotten beyond the theory of numbers and the postulates of geometry. Now, to understand the continuum, we must begin with continuous variation and the conception of the function. In no other way are these ideas brought home to our feeling, our intuition, as through the concepts and the simple laws of natural philosophy, beginning with the geometric problems of mensuration. To think that there is one calculus for the pure mathematician and another for the physicist, the engineer, the geometer, or the cultured layman, is to fail to appreciate that that which is most central in the calculus is its *quantitative* character, through which it measures and estimates the things of the world of our senses. And instruction in the calculus that does not point out—not merely at the beginning or at the end, but all through the course — this close contact with nature, has not done its duty by the student.

Thus we need these problems of natural philosophy in the course in calculus primarily for their own sake, alike for the training of the men who are to go on with mathematics and of those who are not. But there is another reason why we should treat these problems in the course in the calculus, and that is that the teachers in the courses in applied mathematics are frequently ignorant of modern methods in the calculus. Infinitesimals were given to them as small constants, zero, but yet not zero, that can be kept or thrown away at the pleasure of the instructor. They accepted the situation, and if their scientific consciences ever protested, they have long since been silenced. As I said at the beginning, there is a partial truth contained in this archaic view of the infinitesimal, and this is enough for some students. But our colleagues in applied mathematics make a huge mistake when they assume, as mathematicians once did, that their little zeros satisfy their best students. I remember hearing my father relate an amusing anecdote from his college days, which shows how abhorrent this juggling with infinites is to a healthy mind still able to reject the poison. It was in the class in geometry and Professor P. had called on Mr. B. to prove the theorem by which the area of a circle is expressed in terms of the length of its circumference. Mr. B. had learned his lesson from the professor's text-book, in which it was set forth that a circle is a polygon of an infinite number of sides; and he recited the proof down to the conclusion, and there he stopped. "What does that prove?" asked the professor. "It proves an absurdity!" was the reply.

The science has advanced since then. We no longer have the infinite and the infinitesimal constants of former days in mathematics. If we are going to replace them in the courses in applied mathematics by the true infinites and infinitesimals, the variables, we must begin by teaching in the courses in mathematics what these variables are and how to use them.

§ 5. *The Place of the Differential.*

The course in calculus opens with finding the slopes of curves and the derivatives of the elementary functions, and makes application to maxima and minima and to some extent, too, to curve plotting; the more substantial problems in velocities coming a bit later. Here, the limit of the ratio, $\Delta y/\Delta x$, is the central idea, the key to the situation, and the attention of the student should not be embarrassed by any attempt to introduce infinitesimals at this first stage. $\Delta y/\Delta x$ is a function of the independent variable Δx and approaches, in all cases considered, a definite limit (or possibly, for certain values of x_0, becomes infinite) when Δx approaches 0. Its limit should be denoted by an expression that does not suggest a ratio. The expressions of this sort in common use are

$$D_x y, \quad y' \text{ or } y_x, \quad f'(x).$$

But the last three are too meagre a notation for the beginner, and so one naturally chooses $D_x y$.

Practical considerations make it necessary to introduce the differential at an early stage. The student meets the differential notation in his courses in applied mathematics and so it must be explained early in the course in the calculus. My experience, which I notice is confirmed in substance by that of recent writers, leads me to bring in the differential as soon as the differentiation of the algebraic functions has been completed and the applications above mentioned taken up, but while there is still enough drill work in formal differentiation left so that the student will become thoroughly familiar with the method. Since we must teach differentials before the student is ready for them, we may at least get the advantage in technique which accrues from them in such differentiations as

$$d\sqrt{a^2 - x^2} = \frac{1}{2}\,\frac{1}{\sqrt{a^2 - x^2}}\,d(a^2 - x^2) = \text{etc.}$$

The differential is to be introduced as an infinitesimal, a variable, coordinate with the increments Δx, Δy, etc., and differing from its corresponding increment by an infinitesimal of higher order. dy is the principal part of Δy, Δx being taken as the principal infinitesimal. This is the first, but by no means the last word to be said in the course about infinitesimals, and the teacher must make this preliminary exposition brief, but incisive, a careful explanation of the usual geometric figure for illustrating $\Delta x = dx$, Δy, and dy being sufficient for the purpose.

In practice the important side of differentials is the formal side. They form one of the oldest examples of homogeneous variables and their chief value in the calculus consists in their application in formal differentiations and integrations. They appear to good advantage, too, in problems in velocities, for by their aid these problems can be reduced to solving some homogeneous equations in ds, dx, dy, $\cdots$, dt for ds in terms of dt.

§ 6. *The Service of Modern Analysis to the Calculus.*

A question that many of us have asked ourselves is: What service has modern progress in analysis rendered to the calculus of the undergraduate curriculum? At one time it looked as if recent advances in analysis were but to veil the conceptions of the calculus in a mist of ϵ's. But maturer thought disclosed that the formal ϵ, δ-proof is not needed till questions in double limits are reached, and the treatment of these questions belongs to a later stage. The fundamental idea of the limit is that a variable point, restricted according to the conditions of the problem under consideration, ultimately comes and remains within any arbitrarily assigned neighborhood of a certain fixed point, no matter how small this neighborhood may be chosen.

In the development of analysis during the nineteenth century this conception had to meet two demands. The first was that of its adaptability to the needs of the growing science; the second was the refinement which mathematics requires of the ideas which it accepts as fundamental. The fact that it stood both of these tests satisfactorily was an important factor in determining mathematicians to base the calculus on limits, and not on the infinitesimals of Leibniz or the rates of Newton.

A further service of modern analysis lies in its recognition of the elementary theorems about continuity as a secure foundation, a reliable method. If a function $f(x)$, continuous in the interval $a \leqq x \leqq b$, vanishes at the extremities of the interval and is positive at all interior points, or if, more generally, it has at some interior point a value larger than either $f(a)$ or $f(b)$, then it has a maximum value, and in fact a maximum maximorum, within the interval. Likewise, if $f(x)$ becomes positively infinite at each extremity of the above interval and is continuous at all intermediate points, it has a minimum value within the interval. These principles can be used with advantage in the problems in maxima and minima at the beginning of the course. It is important that the student should plot roughly the graphs of the functions he tests for maxima and minima, partly that he may avoid the error of mistaking a relative maximum or minimum at an interior point for a minimum minimorum at an extremity of the interval,* but more especially to give him practice in thinking of a function geometrically as a curve and in reasoning about its properties from inspection of the curve. Nearly all the cases that arise in practice, the "clothed problems," can be dealt with rigorously in this manner without the use of the second derivative, the reasoning being (1) that there is *at least* one maximum or minimum within the interval; (2) that the first derivative of the function (since it exists at each point within the interval) must vanish there; and then it turns out (3) that the latter equation has *only one* root in the interval.

Thus the second derivative is not usually needed for the problems of maxima and minima that arise in practice. It is, however, of value to have an analytic test for relative maxima and minima. Now the most instructive form in which this test can be presented at this early stage in the course is as a problem in curve plotting, and this leads me to say a few words about the right and the wrong way to plot curves.

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* Professor Huntington has given a neat example of this case. A bather standing on the bank of a circular pond, wishes to reach the point directly opposite in the shortest possible time. If he can swim at the rate of u miles an hour and run at the rate of v miles an hour, what point should he swim toward?

We need a few graphs accurately plotted because of the importance of the functions they represent; namely, the graphs of $\sin x$, $\cos x$, $\tan x$, $\log x$ and the inverse functions, and x^n, $n > 0$, $x \geqq 0$. These the student can plot directly from the tables (but he won't do it correctly unless the instructor has good staying powers). What we usually want to know about the graph of a function in the calculus is its character, and even when we need quantitative accuracy, a knowledge of the character of the curve at the outset is a help in plotting. Now the calculus affords a powerful aid for determining the character of a function. If $y = f(x)$ and $D_x y$ is positive, y is increasing with x; if negative, decreasing. Furthermore, if $D_x \tan \tau = D_x^2 y$ is positive, the slope of the curve is increasing with x and so the curve is *concave upward;* if $D_x^2 y$ is negative, the curve is concave downward. In addition to these principles we have the elementary theorems of modern analysis about continuous functions.

That my meaning may be perfectly clear at this point, consider the familiar example:

$$y = x^3 + px + q.$$

Here

$$D_x y = 3x^2 + p, \quad D_x^2 y = 6x.$$

From the last equation we see that for all negative values of x the curve is concave downward, for positive values concave upward. If $q = 0$, the curve is symmetric with respect to the origin; otherwise with respect to the point $x = 0$, $y = q$. Finally, since

$$y \,|_{x=-\infty} = -\infty,$$

we have the character of the curve in its entirety without computing the coordinates of a single point except those of the point of inflection. The slope at that point is p and thus if $p \geqq 0$, the cubic equation

$$x^3 + px + q = 0$$

has but one real root. If $p < 0$, the coordinates of the two maxima and minima tell the story. Other examples of this kind can serve as exercises and thus the graphs of polynomials and rational functions, *if treated by the methods of the calculus above mentioned*, can be used to advantage for training in the calculus.

It is a mistake, however, to dwell very long on these exercises near the beginning of the course. The student learns to apply the above principles by getting examples of their use when he is not looking for them. In one clothed problem or another he will come upon such an equation as

$$\theta - \tan\theta = 0.$$

One root is $\theta = 0$. Is there any further root in the interval from $\theta = 0$ to $\theta = \pi/2$? Let him work out this problem without any suggestions, or only with meagre ones. And when he has finally seen the point, let him find out how many roots the equation has in the interval from $\theta = \pi/2$ to $\theta = 3\pi/2$. Other equations of similar type are:

$$a\theta = \tan\theta \qquad (a \neq 1),$$

$$\theta(1 + \cos\theta) - 2\sin\theta = 0,$$

$$(1 + b^2)\tan^{-1} b = b.$$

And when the orthogonal families of confocal ellipses and hyperbolas are taken up, let him show analytically that the equation in λ:

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1,$$

has one and only one root in each of the intervals $-a^2 < \lambda < -b^2$ and $-b^2 < \lambda < +\infty$.

A simple problem in maxima and minima that calls for curve tracing as here set forth is the following: A gutter whose cross-section is an arc of a circle is to be made by bending into shape a strip of copper. If the width of the strip is a, find the radius of the cross-section when the carrying capacity of the gutter is a maximum.

In the second course in the calculus the Γ-function is introduced by the integral

$$\Gamma(\alpha) = \int_0 x^{\alpha-1}e^{-x}dx \qquad (\alpha > 0),$$

and its derivatives are given by differentiating under the sign of integration. Assuming this process to be justifiable, let the class determine the character of the graph of the function.

Here, then, is another great principle of the calculus. It is not quickly learned. To inculcate it into the student, it is not necessary, however, nor even desirable, to take any large amount of time from other things. What is required is vigilance on the part of the teacher. The psychological moment will not come with all students, or even many, in a given hour. It is necessary to give the class successive opportunities, spread out over the whole course, so that the environment may frequently be favorable for the reception of the desired suggestion. But when a student has once come to incorporate this principle into his mode of thought, he has received from the calculus an aid to that of which I spoke at the beginning — the power to grasp, through the quantitative, the qualitative side of the physical world about us.

§ 7. *Analytic Mechanics.*

A further subject of importance for the training of the undergraduate in mathematics is that of mechanics. This subject is treated most satisfactorily, I believe, in a collateral course in charge of the department of mathematics, provided that the instructor in this course has clear insight into the elementary facts of mechanics and their exemplification in every day life, and is interested primarily in teaching *mechanics*, not analytic geometry and the calculus. It is desirable to begin with the statics of particles and rigid bodies, treated both analytically and graphically. In the statics of a particle the force polygon, drawn to scale, serves as a check on the analytic solution. Half a dozen lectures on the graphical statics of roof and bridge trusses, including the graphical determination of centres of gravity, prove highly interesting to a college class. Thus the student has been clarifying his ideas of force before the dynamics of moving bodies is begun, and in the meantime he has learned in the course in the calculus how to differentiate and integrate. He is now ready for Newton's second law of motion, a desirable foundation for dynamics for the beginner, since it makes him think about inertia and the way a body moves under the action of forces, as well as about the reaction of the parts of a machine on their constraints. The course naturally includes an extended treatment of centrifugal force, the principle of work and energy, and the dynamics of a rigid body in the case of motion in two dimensions. It is also important to set forth the conceptions and the fundamental theorems about the instantaneous center, the space and the body centrode, and the law of cam motion.

From the time dynamics is begun, — about one third of the course should be given to statics, — the calculus is of vital importance. On the other hand the problems, several of which should be worked independently by the student in connection with each lecture, ought not, as a rule, to involve serious difficulties of an analytic nature. There ought to be many easy and some hard exercises; but the difficulty of the latter should be due to the incompleteness of the students mechanical intuition, not to complicated analytic or geometric considerations.*

The course in the calculus ought to contain a chapter in mechanics, in which rectilinear motion under forces that are functions of the space or of the velocity are treated and which thus supplements the course in mechanics without duplicating it. If, however, such a course in mechanics as the above is not given, this chapter must be lengthened; but it cannot in any case replace the course in mechanics. I have for many years given both courses, and I have found the course in mechanics to afford most valuable training in the appreciation of some of the leading principles of the calculus; and this in spite — or perhaps rather because — of the fact that I reduce the analytic difficulties of the course to a minimum.

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* In technical schools the course in graphical statics, if prescribed for all who take this course, makes it possible to abridge slightly the treatment of statics. But the course we are here considering is otherwise independent of that course and the course on the strength of materials.

It is, of course, desirable that a second course in mechanics be given. But inasmuch as a first course of the elementary character here outlined is frequently lacking and is for most students the more instructive of the two, — it is a far more difficult course to give effectively than the advanced course is, — I am concerned here with emphasizing this gap in mathematical instruction.

§ 8. *Advanced Calculus.*

The first course in the calculus, even though it be a five hour course preceded by differentiation in the freshman year, is not adequate for covering all the topics that belong in a proper treatment of the elementary part of the subject. Take, for example, partial differentiation. The easier applications to problems in the geometry of space can be treated in the first course. But the more difficult applications which arise in pure mathematics and in physics, and which are largely formal, clearly should find a place in the second course. Again, in multiple integrals, the conception of the multiple integral, its evaluation by means of iterated integrals, and its use in formulating physical problems belong early, and in technical schools the instruction should be so arranged that all students of engineering will get this part of the subject. But Green's theorem and its applications to exact differentials and to such problems as the deduction of Laplace's equation as the condition for a steady flow of heat or electricity claim an important place in the second course.

Then there are the hyperbolic functions, the elliptic integrals and functions, and the Γ- and B-functions, the latter being defined by definite integrals. When I say that the course should treat thoroughly the simplest tests for the convergence of improper definite integrals, you will all agree with me and some will wonder why I do not demand more, tests for continuity, etc. The reason is that the student is not ready to receive more at this stage. If those of you who give a second course in the calculus will inform yourselves about the student's real power in dealing with the simplest cases where he has to detect unaided whether the integral in question is a proper or an improper integral, you will find a degree of immaturity hard to realize. He will test, for example, the integral

$$\int_0^1 \frac{\log x}{1-x}\, dx$$

for convergence or divergence at the upper end of the interval of integration with the same care that he bestows on the lower end. "Have him plot the graph of the integrand," you say. Good advice; this is an aid. But ask the class to note the time they put on the exercises that are handed in, and take the lesson to heart. The fact is that the student is nowhere near the point where he can even see at a glance in all the simpler cases whether the integrand becomes infinite or remains finite; much less is it easily apparent to him whether to expect convergence or divergence, and what test to apply. Now these are the very things that have to be taught him right here and it takes time. But it is time well spent; it is the teacher's opportunity to strengthen the student's power to deal with functions and at the same time to treat the formal side of the definition of functions by means of definite integrals, the only questions of convergence here considered being those of the mere convergence of the integrals that present themselves. How far beside the mark double limits and ϵ-proofs are at this stage is clear to every teacher who has perceived his opportunity at this point.

The value of a special course on elementary differential equations is generally recognized. But, wholly aside from this course, differential equations should permeate both courses in the calculus. Through them clothed problems of first importance for training in the calculus are introduced.

I have not even mentioned the subjects of the further elementary developments of infinite series, including Fourier's series, of mean value and probability, of the approximate solution of equations, and of a brief introduction to the theory of functions of a complex variable and the calculus of variations. And yet most of these subjects should be studied in an elementary way before the more difficult questions of modern analysis are begun.

This, then, is the elementary material that forms the subject matter of the calculus. To introduce the treatment of questions in double limits at this stage is both to deprive the student of the chance to acquire a broad knowledge of the calculus and to bungle his training in analysis.

§ 9. *Conclusion.*

It has not been my intention to mark out a course in the calculus, for the great body of the instruction at present given by any good teacher is sound and ought to be retained in substance. Before all, there is the necessity of faithful insistence on the part of the instructor on accurate formal work of reasonably substantial character. The technique of the calculus must be taught. Then there is the large body of easy problems in the applications which depart so slightly from the typical problems of the text-books or the class-room that they are within the reach of all who make an honest effort to do the work of the course. Moreover, as regards the choice of material, the important improvement of introducing the integral as the limit of a sum early in the first course may now be fairly regarded as achieved. And yet, with all of this that is so good and sound, if you open one of our text-books on the calculus and ask: What *is* the calculus? What will abide after the formulas are forgotten? What is the soul and the spirit of this great science, as conceived by the man whose work in life does not lie within the field of mathematics? I can't help feeling that the answer does not ring clear: The calculus is the greatest aid we have to the appreciation of physical truth in the broadest sense of the word.

NOTES.

THE fourteenth summer meeting of the AMERICAN MATHEMATICAL SOCIETY will be held at Cornell University on Thursday–Friday, September 5–6. Titles and abstracts of papers to be presented at this meeting should be in the hands of the Secretary by August 24. Abstracts intended to be printed in the announcement of the meeting must be submitted by August 17.

THE April number (volume 8, number 2) of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "Dynamical trajectories: the motion of a particle in an arbitrary field of force," by E. KASNER; "A class of periodic orbits of an infinitesimal body subject to the attraction of n finite bodies," by W. R. LONGLEY; "A proof of some theorems on pointwise discontinuous functions," by E. B. VAN VLECK; "Invariants of binary forms under modular transformations," by L. E. DICKSON; "Projective differential geometry of curved surfaces (first memoir)," by E. J. WILCZYNSKI; "A method for constructing the fundamental region of a discontinuous group of transformations," by J. I. HUTCHINSON; "Oblique reflections and unimodular strains," by E. B. WILSON; "On the introduction of convergence factors into summable series and summable integrals," by C. N. MOORE.

THE pages of "New Publications" contained in each number of the BULLETIN will hereafter be reprinted and furnished to such institutions as may desire to use them for the purpose of ordering mathematical books. Enquiries should be addressed to the Committee of Publication.

ON the editorial staff of the *Annals of Mathematics* Professor J. K. WHITTEMORE will succeed Professor MAXIME BÔCHER, who is about to assume his duties as member of the Editorial Committee of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY.

AT the meeting of the London mathematical society held on April 11, the following papers were read: By H. F. BAKER, "A theorem in the theory of functions"; by H. BATEMAN, "An introduction to the metrical geometry of space of n dimensions," and "The values of the parameters for which a definite integral can be zero"; by H. HILTON, "A note on Perott's theorem"; by E. W. HOBSON, "Poisson's integral and its relations to the proof of Fourier's theorem."

THE supplementary announcement of the annual list of members of the Circolo matematico di Palermo issued in March shows that the society now has 461 active members, of whom 75 are Americans. This society was organized in the spring of 1884 by residents of Palermo; it at once became national and soon international in its scope. At present 288 of its members are foreigners. The regular sessions of the society are held on alternate Sunday afternoons. The first volume of the *Rendiconti* was issued in 1887; at least one volume has appeared each succeeding year, and during each of the years 1905 and 1906 two volumes were published. The parts appear bimonthly; they contain brief reports of the sessions and such memoirs read before the society as are appropriate. Beginning with January, 1907, the reports were transferred to the *Supplemento*, which also gives the titles of papers published in other mathematical journals. The *Annuario* contains the list of members and a complete index of all papers published in the *Rendiconti*. In the exchange list of the society are 105 periodicals, including all the important mathematical journals of the world. The society, together with the Accademia dei Lincei at Rome, will have charge of the arrangements for the fourth international congress of mathematicians to be held at Rome next April. The present officers of the Circolo are: president, Professor M. L. ALBEGGIANI; secretaries, Professors G. DI SIMONE and E. P. GUERRA; director of the *Rendiconti*, Professor G. B. GUCCIA.

ON the occasion of the Abel centenary celebration it was decided to erect a memorial tablet to his honor, and funds were subscribed for the purpose. This plan has now been succeeded by another, for which larger means are needed. A plaster cast of a monument 40 feet high representing Abel supported by gigantic allegorical figures has been prepared, which is to be cast in bronze. The work has been done by Gustav Vigeland, perhaps the ablest sculptor in Norway. In order that this undertaking may assume an international character, mathematicians throughout the world are invited to contribute to it. The circular of invitation bears the names of a responsible committee. Contributions may be sent to Professor L. Sylow, University of Christiania, Norway.

THE following advanced courses in mathematics are offered during the academic year 1907–1908:

COLUMBIA UNIVERSITY.—By Professor F. N. COLE: Theory of groups, three hours; Theory of invariants, three hours.—By Professor JAMES MACLAY: Application of the calculus to the theory of surfaces and curves in space, three hours; Theory of functions of a complex variable, three hours.—By Professor C. J. KEYSER: Modern theories in geometry, three hours; General theory of assemblages, three hours.—By Professor H. B. MITCHELL: Vector analysis, two hours; Differential equations, two hours.—By Professor EDWARD KASNER: Differential equations and continuous groups, three hours; General introduction to higher mathematics, three hours.—By Dr. G. H. LING: Modern higher algebra, three hours.—By Professor M. I. PUPIN: Theory of the potential function, two hours; Hydrokinetics, two hours; Partial differential equations of physics, two hours; Special problems, two hours.—By Professor A. P. WILLS: Mechanics, two hours; Theory of elasticity, two hours; Electricity and magnetism, electro-magnetic theory of light, two hours; Thermodynamics, two hours.

CORNELL UNIVERSITY. — By Professor L. A. WAIT: Advanced analytic geometry, three hours. — By Professor J. M. MAHON: Theory of potential and spherical harmonics, three hours. — By Professor J. H. TANNER: Theory of equations, two hours; Teachers' course in algebra, two hours. — By Professor J. I. HUTCHINSON: Automorphic functions, three hours. — By Professor V. SNYDER: Algebraic curves, three hours. — By Professor W. B. FITE: Advanced calculus, three hours; Theory of functions of a real variable, two hours. — By Dr. F. R. SHARPE: Theory of electrons, three hours. — By Dr. W. B. CARVER: Projective geometry, three hours. — By Dr. A. RANUM: Differential equations, two hours. — By Dr. D. C. GILLESPIE: Advanced differential equations, two hours. — By Mr. C. F. CRAIG: Theory of probabilities and insurance, two hours. The Oliver mathematical club will meet weekly.

JOHNS HOPKINS UNIVERSITY. — By Professor F. MORLEY: Vector analysis, two hours; Higher geometry, two hours; Seminar, one hour; Classic authors, one hour. — By Dr. A. COHEN: Differential equations, two hours; Elementary theory of functions, two hours; Introduction to differential equations and vector analysis, two hours. — By Dr. A. B. COBLE: Cremona transformations, two hours (first half year); Theory of statistics, two hours (second half year).

UNIVERSITY OF PENNSYLVANIA. — By Professor E. S. CRAWLEY: Solid analytic geometry, two hours; Higher plane curves, three hours. — By Professor G. E. FISHER: Theory of functions of a complex variable, first half year, three hours; Elliptic functions, second half year, three hours. — By Professor I. J. SCHWATT: Definite integrals, three hours. — By Professor G. H. HALLETT: Lies theory of continuous groups, first half year, three hours; Galois theory of algebraic equations, second half year, three hours. — By Dr. F. H. SAFFORD: Curvilinear coördinates, three hours. — By Dr. O. E. GLENN: Higher algebraic equations, two hours.

AT the meeting of the National academy of sciences held in Washington April 16 and 17, Professor A. R. FORSYTH, of Cambridge University, and Professor D. HILBERT, of the University of Göttingen were elected to associate membership.

PROFESSOR J. TANNERY, of the University of Paris, has been elected a member of the Paris academy of sciences.

PROFESSOR A. LOEWY, of the University of Freiburg, has been commissioned to give regular courses in the mathematics of insurance.

PROFESSOR G. DARBOUX has been appointed associate member of the Belgian academy of sciences.

PROFESSOR PH. FURTWÄNGLER, of the agricultural academy at Poppelsdorf has been appointed professor of mathematics at the technical school of Aachen.

PROFESSOR S. GUNDELFINGER, of the technical school at Darmstadt, retired from active service April 1, on account of ill health.

DR. W. LUDWIG, of the technical school at Karlsruhe, has been appointed professor of descriptive geometry at the technical school of Braunschweig.

PROFESSOR E. VON WEBER, of the University of Munich, has been appointed associate professor of mathematics at the University of Würzburg.

PROFESSOR K. WEIGHARDT, of the technical school at Braunschweig, has been appointed professor of mathematics at the technical school of Hanover.

DR. P. KOEBE has been appointed docent in mathematics at the University of Göttingen.

AT McGill University, W. M. EDWARDS, lecturer in mathematics, has been made assistant professor of civil engineering.

PROFESSORS T. S. FISKE and D. E. SMITH, of Columbia University, will be abroad on leave of absence during the academic year 1907–08.

AT the University of Nebraska, Professor A. L. CANDY has been promoted to a full professorship of mathematics and Professor C. C. ENGBERG to a full professorship of applied mathematics.

DR. S. D. TOWNLEY, astronomer in charge of the International Latitude Observatory at Ukiah, California, and lecturer in astronomy in the University of California, has been appointed to an assistant professorship in the department of applied mathematics at Stanford University.

DR. G. D. BIRKHOFF has been appointed instructor in mathematics at the University of Wisconsin.

MR. W. M. CARRUTH, of Cornell University, has been appointed assistant professor of mathematics at Hamilton College.

AT the Massachusetts Institute of Technology the following appointments have been made: Dr. E. B. WILSON, of Yale University, associate professor of mathematics; Dr. H. B. PHILLIPS, of the University of Cincinnati, and Mr. N. J. LENNES, of the University of Chicago, instructors in mathematics. Dr. C. L. E. MOORE has been granted a leave of absence and will spend next year studying in Italy and in Germany.

PROFESSOR MANSFIELD MERRIMAN has retired from the professorship of civil engineering at Lehigh University after a record of twenty-eight years' service.

CHARLES H. HINTON, author of several popular works on the fourth dimension and of late an examiner in the Patent Office at Washington, died in that city on April 30, 1907, at the age of 63 years.

DR. A. FUHRMANN, emeritus professor of mathematics at the Dresden polytechnic school, well known for his work on mechanics, died April 30 at the age of 67 years.

PROFESSOR FERDINANDO ASCHIERI, of the University of Pavia, died April 14 at the age of 60 years.

THE death has recently been announced at the age of 91 years of Professor M. Steinschneider, of Berlin, one of the best authorities on the history of Hebrew mathematics. Professor Steinschneider published numerous works on Hebrew and Hebrew-Arabic literature, including catalogues of the Hebrew manuscripts in various large libraries.

CATALOGUES of second-hand mathematical works: Gustav Fock, Schlossgasse 7, Leipzig, catalogue No. 300, 3,255 titles in mathematics and natural science; K. F. Kohler's Antiquarium, Kurprinzstrasse 6, Leipzig, catalogue No. 569, periodicals and proceedings of societies, about 2,000 titles; Martin Boas, Karlstrasse 25, Berlin, N. W., catalogue No. 60, astronomy, mathematics and physics; H. Lüneburg, Karlstrasse 4, Munich, catalogue No. 73, about 1900 titles in mathematics, astronomy, and physics, including works from the library of the late Professor G. Bauer.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

CAJORI (F.). See CANTOR (M.).

CANTOR (M.). Vorlesungen über Geschichte der Mathematik. Herausgegeben von M. Cantor, unter Mitwirkung von V. Bobynin, A. von Braunmühl, F. Cajori, u. a. Vol. IV: Von 1759–1799. Lieferung 1: XIX. Abschnitt: S. Günther, Geschichte der Mathematik; XX. Abschnitt: F. Cajori, Arithmetik, Gleichungslehre, Zahlentheorie. Leipzig, Teubner, 1907. 8vo. 200 pp. M. 5.60

——. Vorlesungen über Geschichte der Mathematik. Band I: Von den ältesten Zeiten bis zum Jahr 1200 nach Christus. 3te Auflage. Leipzig, Teubner, 1907. 8vo. 6 + 941 pp., 1 plate. M. 24.00

DU PASQUIER (L. G.). Zahlentheorie der Tettarionen. (Diss.) Zürich, 1906. 8vo. 75 pp.

FLECHSIG. See LUCZAC.

FORSYTH (A. R.). Theory of differential equations. Part IV: Partial differential equations. 2 vols. Cambridge, 1907. 8vo. Cloth.

GÜNTHER (S.). See CANTOR (M.).

HALBWACHS (M.). Leibniz. Paris, Delaplane, 1906. 18mo. 124 pp. F. 0.90

LAISANT (C. A.). La mathématique. Philosophie. Enseignement. 2e édition, revue et corrigée. Paris, Gauthier-Villars, 1907. 8vo. 7 + 244 pp. Boards. F. 5.00

LARMOR (J.) See STOKES (G. G.).

MARC (L.). Aufgaben aus der höheren Mathematik, technischen Mechanik und darstellenden Geometrie, welche bei der Vorprüfung für Bau-, Maschinen-, Elektro-, Kultur- und Vermessungs-Ingenieure sowie Architekten an der k. technischen Hochschule zu München vom Jahr 1901 ab gestellt worden sind. München, Lachner, 1907. 8vo. 48 pp. M. 2.00

ROMANO (F.). Sopra una trasformazione doppia del terzo ordine nei punti dello spazio. Avola, Piazza, 1906. 8vo. 47 pp.

SISAM (C. H.). On septic scrolls having a rectilinear directrix. (Diss., Cornell University.) Baltimore, 1907. 4to. 52 pp. (*American Journal of Mathematics*, Vol. 29, pp. 48–100.)

SOMMER (J.). Vorlesungen über Zahlentheorie. Einführung in die Theorie der algebraischen Zahlkorper. Leipzig, Teubner, 1907. 8vo. 6 + 361 pp. Cloth. M. 6.00

STOKES (G. G.). Memoirs and scientific correspondence of Sir George Gabriel Stokes. Selected and arranged by J. Larmor. 2 vols. Cambridge, 1907. 8vo. 1004 pp. 4 plates. Cloth.

THAER (C.). Ueber Invarianten, die symmetrischen Eigenschaften eines Punktsystems entsprechen. Leipzig, Teubner, 1906. 8vo. 31 pp. M. 0.80

VOGT (H.). Eléments de mathématiques supérieures, à l'usage des physiciens, chimistes et ingénieurs et des élèves des Facultés des sciences. 4e édition, très augmentée et entièrement refondue. Paris, Vuibert, 1907. 8vo. 8 + 712 pp.

VOGT (W.). Korrelative Räume bei gegebener Punktkernflache. (Diss.) Breslau, 1906. 8vo. 69 pp.

WHITE (M. B.). The asymptotic lines on the anchor ring. (*Annals of Mathematics*, April, 1907.) 4to. 14 pp.

WIESING (O.). Ueber eine zwei-zweideutige Verwandtschaft zwischen zwei Ebenen und ihr Analogon im Raume. (Diss.) Breslau, 1906. 8vo. 83 pp.

II. ELEMENTARY MATHEMATICS.

DURELL (F.). Key to plane and solid geometry. New York, Maynard, 1906. 12mo. 173 pp. Cloth. $1.00

GORSE (F.). A school algebra course. Cambridge, University Press, 1907. 12mo. 312 pp. Cloth. 3s.

HAMMER (E.). Handbuch der ebenen und spharischen Trigonometrie. Zum Gebrauch beim Selbstunterricht und in Schulen, besonders als Vorbereitung auf Geodäsie und sphärische Astronomie. Stuttgart, Metzler, 1907. 3te, erweiterte Auflage. 8vo. 18 + 644 pp. M. 10 60

JACKSON (W. H.). Elementary solid geometry, including the mensuration of the simpler solids. London, Arnold, 1907. 12mo. 172 pp. 2s. 6d.

JACQUET (E.) et LACLEF (A.). Compléments d'arithmétique, d'algèbre, de géométrie, accompagnés de très nombreux problèmes donnés dans les examens, spécialement rédigés à l'usage des candidats aux écoles des arts et métiers. Paris, Nathan, 1907. 16mo. 388 pp.

LACLEF (A.). See JACQUET (E.).

MENGER (J.). Leitfaden der Geometrie fur Gewerbeschulen. 4te Auflage. Wien, Hölder, 1907. 8vo. 4 + 77 pp. Boards. M. 0.96

MINET (A.) et PATIN (L.). Cours pratique d'arithmétique, de système métrique et de géométrie. Paris, Nathan.

MINET (A.) et PATIN (L.). Cours élémentaire (1re et 2e années) 1906. 16mo. 240 pp. F. 2.25

——. Cours moyen. 1907. 16mo. 288 pp. F. 1.25

MORF. See TZAUT (S.).

PATIN (L.). See MINET (A.).

SUPLEE (H. H.). Five figure logarithms. Philadelphia, Lippincott, 1906. 16mo. Leather. $2.00

TZAUT (S.) et MORF. Exercices et problèmes d'algèbre, et réponses. (Première série.) 4e édition. Paris, Gauthier-Villars, 1906. 16mo. F. 5.00

III. APPLIED MATHEMATICS.

ADAMS (C. L.). Notes on descriptive geometry. Part I. Prepared for use in Massachusetts Institute of Technology. Boston, Adams, 1907. 8vo. Cloth. $1.50

ALBRECHT (M. F.) und BIEROW (C. S.). Lehrbuch der Navigation und ihrer mathematischen Hilfswissenschaften. Fur die k. preussischen Navigations-Schulen. 9te Auflage. Berlin, Decker, 1906. 8vo. 22 + 528 pp. M. 14.00

BARTLETT (G. M.). Numerical problems in descriptive geometry, for class and drawing room practice. Revised edition. Ann Arbor, Bartlett, 1907. 83 pp. Cloth. $0.50

BENISCHKE (G). Die wissenschaftlichen Grundlagen der Elektrotechnik. 2te erweiterte Auflage von "Magnetismus und Elektrizität mit Rücksicht auf die Bedurfnisse der Praxis." Berlin, Springer, 1907. 8vo. 15 + 580 pp. M. 12.00

BIEROW (C. S.). See ALBRECHT (M. F.).

BROGGI (V.) Matematica attuariale: teoria statistica della mortalità; matematica delle assicurazioni sulla vita. Milano, Hoepli, 1906. 15 + 347 pp. L. 3 50

CHAMPION (E). Principes généraux d'électricité théorique et pratique, à l'usage des élèves des écoles industrielles Vol. III: Dynamos et moteurs à courant continu; alternateurs monophasés et polyphasés; moteurs synchrones; transformateurs. Lille, Danel, 1907. 4to. 152 pp.

FISCHER (O.) Kinematik organischer Gelenke. Braunschweig, Vieweg, 1907. 8vo. 12 + 261 pp. (Die Wissenschaft; Sammlung naturwissenschaftlicher und mathematischer Monographien, Heft 18.) M. 8.00

GENOVINO (G.). Influenza dell'attrazione del sole e della luna sulla direzione della verticale. sulla gravità e sulla marcia dei pendoli. Firenze, Landi, 1907. 8vo. 64 pp.

HELLER (A. H.). Stresses in structures and the accompanying deformations. Columbus, Green, 1907. 8vo. 324 pp. Cloth. $4.00

JÄGER (G.). Theoretische Physik. Band II: Licht und Wärme. Band III: Elektrizität und Magnetismus. 3te verbesserte Auflage. Leipzig, Göschen, 1907. 16mo. 302 pp. (Sammlung Göschen.) M. 1.60

JORDAN (W.). Handbuch der Vermessungskunde. Band III: Landes-Vermessung und Grundaufgaben der Erd-Messung. 5te erweiterte Auflage. Bearbeitet von C. Reinhertz mit Vorwort von E. Hammer. Stuttgart, Metzler, 1907. 8 + 678 pp. M. 15.00

LIAPOUNOFF (A.). Sur les figures d'équilibre peu différentes des ellipsoides d'une masse liquide homogène douée d'un mouvement de rotation. Partie I: Etude générale du problème. St. Petersbourg, 1907. 4to. 225 pp. F. 7.00

MALCOLM (C. W.). Elements of graphic statics. Champaign, Malcolm, 1907. 8vo. 3 + 127 pp. Cloth. $1.75

MATRICULATION mechanics papers. London, Clive, 1907. 12mo. 1s. 6d.

PIERS (F.). Piers' mathematics for the machine shop: a pocket book for the use of machinists and tool makers. Philadelphia, Piers, 1906. 16mo. 78 pp. $1.00

RICHARD (G.). Mécanique. A l'usage des ingénieurs, constructeurs-mécaniciens, industriels, chefs d'ateliers et contremaîtres. 29e édition. Paris, Dunod et Pinat, 1907. 16mo. 5 + 197 pp. Fr. 2.50

VALLEREY (J.). Géométrie et mécanique. 2e édition, revue et corrigée conformément aux programmes. Paris, Challamel, 1907. 8vo. 522 pp. F. 7.50

MODULAR THEORY OF GROUP CHARACTERS.

BY PROFESSOR L. E. DICKSON.

1. THE problem of the representation of a given finite group as a linear homogeneous group with real or complex coefficients has been fully treated by Frobenius * by means of his theory of group characters. The present paper and the companion paper to appear simultaneously in the *Transactions* give a first attack on the corresponding problem for linear congruence groups, and in general for finite linear groups in any field F having a prime modulus p. To obtain simple results, it is in general necessary to introduce certain irrationalities, viz., roots of equations with coefficients in F. As our reference field we shall take the field F_p composed of the totality of integral rational functions with integral coefficients of all Galois imaginaries of all degrees, *i. e.*, the roots of congruences irreducible modulo p. In other words, F_p is the aggregate of the Galois fields $GF[p^n]$, $n = 1, 2, 3, \cdots$. Hence every equation with coefficients in F_p is completely solvable in F_p.

The paper also gives a report on the various expositions of the algebraic theory from the standpoint of their availability in the treatment of the modular theory (cf. §§ 3, 5, but particularly § 13).

2. *Definitions.* Given a finite group H with the h elements $H_0, H_1, \cdots, H_{h-1}$, we shall say that the h matrices of degree f (or linear substitutions)

$$(1) \qquad A_{H_i} = (a^{H_i}_{\alpha\beta})_{\alpha,\beta=1,\cdots,f} \quad (i = 0, 1, \cdots, h-1),$$

whose elements a are marks of the field F_p, define a representation of the group H if the matrices satisfy the h^2 relations

$$(2) \qquad A_R A_S = A_{RS} \quad (R, S = H_0, \cdots, H_{h-1}).$$

The matrices need not be distinct, so that the isomorphism may be multiple. Let $x_{H_i} (i = 0, \cdots, h-1)$ be independent variables. Then

$$(3) \qquad X = \sum_R A_R x_R \quad (R = H_0, \cdots, H_{h-1})$$

is called the group matrix corresponding to the representation.

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* *Berliner Sitzungsberichte*, from 1896 to date.

If B is a matrix of degree f whose elements are marks of F_p and of determinant not zero, then $B^{-1}XB$ is also a group matrix, called equivalent to X. A group matrix X is called reducible or irreducible (in F_p), according as it is or is not equivalent to a matrix $\left(\begin{smallmatrix}Y & O\\ W & Z\end{smallmatrix}\right)$, where Y and Z are square matrices, W a rectangular matrix.

The system of h marks $\chi(H_0), \cdots, \chi(H_{h-1})$, defined by

$$\chi(R) = \sum_{\alpha=1}^{f} a_{\alpha\alpha}^{R} \qquad (R = H_0, \cdots, H_{h-1}), \tag{4}$$

is called a character of the group H for the field F_p, corresponding to the particular representation (1) or group matrix (3). If the latter is irreducible, the character is called simple.

Analogous to the multiplication table of H, the matrix

$$(x_{PQ^{-1}}) \qquad (P,\ Q = H_0, \cdots, H_{h-1}), \tag{5}$$

in which P indicates the row and Q the column, is called the regular group matrix.

3. For the reasons to be pointed out in § 13, the only one of the various expositions of Frobenius's theory which may be utilized in the construction of a corresponding general modular theory is that by I. Schur, "Neue Begründung der Theorie der Gruppencharaktere," *Berliner Sitzungsberichte*, March 23, 1905. In Frobenius's theory, matrix (5) is completely reducible (with zero matrices to the right and left of the irreducible diagonal matrices), and each irreducible factor occurs to a power equal to its degree; in the modular theory these theorems do not hold true when p divides the order h of the group. In the latter case, only the earlier part of Schur's work can be utilized for the modular theory and then only after essential modifications.

The developments by Schur, pages 409–411, as well as the auxiliary theorems from the general theory of matrices, are valid for our field F_p. His theorem I is valid in any field; his theorem II, however, is valid only in a field F which, like F_p or the field of all real and complex numbers, has the property that every equation with coefficients in F is solvable in F. Instead of his fundamental theorem IV, we have the following (valid whether or not p divides f):

THEOREM. *If in the field F_p X and X' are non-equivalent irreducible group matrices of degrees f and f',*

(6) $X=(x_{\alpha\beta}),\ x_{\alpha\beta}=\sum_R a^R_{\alpha\beta}x_R(\alpha,\beta=1,\cdots,f;\ R=H_0,\cdots,H_{h-1}),$

(7) $X'=(x'_{\alpha\beta}),\ x'_{\alpha\beta}=\sum_R b^R_{\alpha\beta}x_R(\alpha,\beta=1,\cdots,f';\ R=H_0,\cdots,H_{h-1}),$

then the following relations hold:

$$\text{(I)} \qquad \sum_R a^{R^{-1}}_{\alpha\beta}a^R_{\gamma\delta}=ce_{\alpha\delta}e_{\beta\gamma} \qquad (\alpha,\beta,\gamma,\delta=1,\cdots,f),$$

$$\text{(II)} \qquad \sum_R a^{R^{-1}}_{\alpha\beta}b^R_{\gamma\delta}=0 \qquad (\alpha,\beta=1,\cdots,f;\ \gamma,\delta=1,\cdots,f'),$$

in which $e_{\alpha\delta}=0$ if $\alpha \neq \delta$, $e_{\alpha\alpha}=1$, while c is a constant for a given group matrix X, equal to that for any equivalent group matrix, and

$$\text{(8)} \qquad fc=h.$$

In view of Schur's proof we have (II) and

$$\text{(I}_1\text{)} \qquad \sum_R a^{R^{-1}}_{\alpha\beta}a^R_{\gamma\delta}=c_{\beta\gamma}e_{\alpha\delta},$$

in which the $c_{\beta\gamma}$ are constants to be determined. In view of (1), relations (2) are equivalent to

$$\text{(9)} \quad \sum_{\beta=1}^{f} a^R_{\alpha\beta}a^S_{\beta\delta}=a^{RS}_{\alpha\delta} \quad (\alpha,\delta=1,\cdots,f;\ R,S=H,\cdots,H_{h-1}).$$

In (I$_1$), replace α by ρ, γ by σ; then multiply by $a^S_{\alpha\rho}a^T_{\gamma\sigma}$ and sum for $\rho,\sigma=1,\cdots,f$. We get

$$\sum_R\Big(\sum_\rho a^S_{\alpha\rho}a^{R^{-1}}_{\rho\beta}\Big)\Big(\sum_\sigma a^T_{\gamma\sigma}a^R_{\sigma\delta}\Big)=\sum_{\rho,\sigma}e_{\rho\delta}c_{\beta\sigma}a^S_{\alpha\rho}a^T_{\gamma\sigma}.$$

Applying (9) we get

$$\text{(10)} \qquad \sum_R a^{SR^{-1}}_{\alpha\beta}a^{TR}_{\gamma\delta}=a^S_{\alpha\delta}\sum_\sigma c_{\beta\sigma}a^T_{\gamma\sigma}.$$

For $T=E=$ identity, we get, since $a^E_{\gamma\sigma}=e_{\gamma\sigma}$ by Schur's proof,

$$\text{(11)} \qquad \sum_R a^{SR^{-1}}_{\alpha\beta}a^R_{\gamma\delta}=a^S_{\alpha\delta}c_{\beta\gamma}.$$

In (10) replace $c_{\beta\sigma}$ by its value from (I$_1$) for $\alpha=\delta$, $\gamma=\sigma$. The right member of (10) becomes

$$a^S_{\alpha\delta}\sum_\sigma a^T_{\gamma\sigma}\Big(\sum_R a^{R^{-1}}_{\alpha\beta}a^R_{\sigma\alpha}\Big)=a^S_{\alpha\delta}\sum_R a^{R^{-1}}_{\alpha\beta}a^{TR}_{\gamma\alpha},$$

in view of (9). In (11) replace R by R^{-1}, S by T, γ by α, δ by β, α by γ, β by α. We get

$$\sum_R a^{R^{-1}}_{\alpha\beta} a^{TR}_{\gamma\alpha} = a^T_{\gamma\beta} c_{\alpha\alpha}.$$

We have now shown that (10) becomes

$$\text{(III)} \qquad \sum_R a^{SR^{-1}}_{\alpha\beta} a^{TR}_{\gamma\delta} = a^S_{\alpha\delta} a^T_{\gamma\beta} c_{\alpha\alpha}.$$

In (III) set $T = E$ and compare the result with (11). We get

$$\text{(12)} \qquad a^S_{\alpha\delta} e_{\gamma\beta} c_{\alpha\alpha} = a^S_{\alpha\delta} c_{\beta\gamma} \quad (\alpha,\ \beta,\ \gamma,\ \delta = 1, \cdots, f).$$

If for α fixed, the $a^S_{\alpha\delta}$ were all zero, then by (6) every $x_{\alpha\beta} = 0$, *i. e.*, all the elements of the αth row of X would vanish. But $|X|$ is not identically zero. Hence, by (12),

$$c_{\beta\gamma} = e_{\beta\gamma} c_{\alpha\alpha} \qquad (\alpha,\ \beta,\ \gamma = 1, \cdots, f).$$

In particular, the $c_{\alpha\alpha}$ are all equal. Call their common value c. Then

$$\text{(13)} \qquad c_{\beta\gamma} = c e_{\beta\gamma}.$$

Hence (I_1) becomes (I). In (II) replace α by ρ, γ by σ; then multiply by $a^S_{\alpha\rho} b^T_{\gamma\sigma}$ and sum for ρ, σ. We get

$$\text{(IV)} \qquad \sum_R a^{SR^{-1}}_{\alpha\beta} b^{TR}_{\gamma\delta} = 0.$$

Now (I), for $\delta = \alpha$, $\gamma = \beta$, becomes

$$\text{(14)} \qquad c = \sum_R a^{R^{-1}}_{\alpha\beta} a^R_{\beta\alpha}.$$

Summing for β, applying (9) and $a^E_{\alpha\alpha} = 1$, we obtain (8).

Finally, we prove that $c = d$, d being the constant for

$$P^{-1}XP = \sum_R (d^R_{\alpha\beta}) x_R.$$

Let $P = (p_{\alpha\beta})$, $P^{-1} = (q_{\alpha\beta})$, so that

$$S_\delta \equiv \sum_\sigma q_{\delta\sigma} p_{\sigma\delta} = 1.$$

Then

$$d^R_{\gamma\delta} = \sum_{\rho,\ \sigma} q_{\gamma\rho} a^R_{\rho\sigma} p_{\sigma\delta},$$

$$d \equiv \sum_R d^{R^{-1}}_{\gamma\delta} d^R_{\delta\gamma} = \sum_{\rho,\ \sigma,\ r,\ s} q_{\gamma\rho} p_{\sigma\delta} q_{\delta r} p_{s\gamma} \Big(\sum_R a^{R^{-1}}_{\rho\sigma} a^R_{rs} \Big).$$

The final term equals $ce_{\rho\delta}e_{\sigma\tau}$ by (I). Hence

$$d = c\Big(\sum_\rho q_{\gamma\rho}p_{\rho\gamma}\Big)\Big(\sum_\sigma q_{\delta\sigma}p_{\sigma\delta}\Big) = cS_\gamma S_\delta = c.$$

4. Certains relations between the simple characters (4) follow readily from (I)–(IV) and (9). In (III) take $\beta = \alpha$, $\delta = \gamma$, and sum for α, γ. Hence

$$\text{(V)} \qquad \sum_R \chi(SR^{-1})\chi(TR) = c\chi(ST).$$

For $T =$ identity, (V) becomes

$$\text{(VI)} \qquad \sum_R \chi(SR^{-1})\chi(R) = c\chi(S).$$

Conversely, replacing S by ST and R by RT in (VI), we get (V). In fact, by (4) and (9), we find that

$$\text{(VII)} \qquad \chi(RT) = \chi(TR), \quad \chi(E) = f \quad (E = \text{identity}).$$

For $S =$ identity, (VI) becomes, by (8),

$$\text{(VIII)} \qquad \sum_R \chi(R^{-1})\chi(R) = h.$$

In (III) take $\gamma = \beta$, $\delta = \alpha$, sum for α, β and apply (9). Hence

$$\text{(IX)} \qquad c\chi(S)\chi(T) = \sum_R \chi(SR^{-1}TR).$$

Let χ' be the character corresponding to the irreducible group matrix X' not equivalent to X. In (IV) set $\beta = \alpha$, $\delta = \gamma$, and sum for α, γ. Hence

$$\text{(X)} \qquad \sum_R \chi(SR^{-1})\chi'(TR) = 0.$$

If the modulus p does not divide h, relations (VI) and (X) follow* from (VII), (VIII), (IX) (the latter three with $c = h/f$ serve in the algebraic theory to determine completely the characters).

Further, when p is prime to h, the entire exposition by Schur is valid in the field F_p. In particular the algebraically irreducible factors of the regular group determinant have integral algebraic numbers as coefficients and hence are functions interpretable in F_p and are irreducible in F_p. The fact that the modular theory for p prime to h is identical with the algebraic theory was first stated in the writer's paper, *Transactions*, volume 3 (1902), page 285.

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* Cf. Weber, Algebra, 2d ed., II, p. 194.

5. For a given group H of order h, let X, X', $\cdots$ be group matrices of degrees f, f', $\cdots$, irreducible in the field F_p, and no two equivalent. Give them the notation (6), (7), etc. Then the $f^2+f'^2+\cdots$ linear functions $x_{\alpha\beta}$, $x'_{\alpha\beta}$, $\cdots$ are linearly independent in F_p.

In case p does not divide h, the proof follows at once from relations (I) and (II) (cf. Schur, page 412). In the contrary case the proof fails, since c vanishes for certain group matrices. We may, however, make use of theorem I in the paper by Frobenius and Schur on the equivalence of linear groups, *Berliner Sitzungsberichte*, 1906, page 209, the proof being valid in our field F_p.

It follows that the determinant of an irreducible group matrix X is an irreducible function of the variables x_R; and that two irreducible group matrices are equivalent if and only if their determinants are identically equal.

Let Φ denote the determinant $|X|$ of the irreducible group matrix (3) of degree f. If E is the identity element of the group, and $R \neq E$, the coefficient of $x_E^{f-1}x_R$ in Φ is $\chi(R)$, in view of (4) and $a_{\alpha\beta}^E = e_{\alpha\beta}$. Also $\chi(E) = f$ by (VII). Hence for every element R, $\chi(R)$ is the coefficient of x_E^{f-1} in $\partial\Phi/\partial x_R$. This property is taken as the definition of $\chi(R)$ in Frobenius's second exposition, *Sitzungsberichte*, 1896, page 1349. The latter paper gives a method of determining all the coefficients of Φ in terms of the characters $\chi(R)$. The method must be modified in the case of a modular field. If u is a new independent variable, set

$$(15) \qquad \Phi(x_E+u, x_A, x_B, \cdots) = u^f + \Phi_1 u^{f-1} + \cdots + \Phi_f,$$

where Φ_n is an integral homogeneous function of the nth degree of x_E, x_A, $\cdots$, and $\Phi_f = \Phi$. Let S_n denote the sum of the nth powers of the negatives of the quantities $u_1, \cdots, u_f$ for which (15) vanishes. By Frobenius's proof,

$$S_1 = \Phi_1 = \sum_R \chi(R) x_R, \quad S_n = \sum_{R_1, \ldots, R_n} \chi(R_1 \cdots R_n) x_{R_1} \cdots x_{R_n},$$

where $R_1, \cdots, R_n$ range independently over the h elements of

H. By Waring's formula each Φ_t is expressed as a polynomial in the S_n. For a modular field, this formula is not always applicable in view of the denominators. We shall illustrate for small values of f a valid method of determining in F_p the function

$$(16)\qquad \Phi=\sum C_{R_{i_1},\cdots,R_{i_f}}x_{R_{i_1}}\cdots x_{R_{i_f}}\qquad (i_1\leqq i_2\leqq\cdots\leqq i_f).$$

For $f=2$, Φ is determined by

$$(17)\qquad 2\Phi_2=S_1^2-S_2,$$

unless $p=2$. By comparing the coefficients of $x_E y_E x_R y_S$ $(R\neq E,\ S\neq E)$ in

$$(18)\qquad \Phi(z)=\Phi(x)\Phi(y),\quad z_T=\sum_R x_R y_{R^{-1}T},$$

we obtain,* according as $R\neq S$ or $R=S$,

$$(19)\quad C_{RS}=\chi(R)\chi(S)-\chi(RS),\quad 2C_{RR}=\chi(R)\chi(R)-\chi(R^2).$$

For modulus $p=2$, it remains to determine C_{RR}. Now

$$-S_3+\Phi_1 S_2-\Phi_2 S_1=0,$$

by Newton's identities. The coefficients of

$$x_R^3,\ x_R^2 x_S\quad (R\neq S\neq E),$$

give

$$C_{RR}\chi(R)=\chi(R)\chi(R^2)-\chi(R^3),$$

$$C_{RR}\chi(S)=-C_{RS}\chi(R)-3\chi(R^2S)+\chi(R^2)\chi(S)+2\chi(RS)\chi(R).$$

But C_{RS} is given by (19_1). Hence C_{RR} is determined by the characters unless the latter all vanish. But in that case Φ involves only the squares of the variables x_R and hence is reducible in F_2, contrary to hypothesis.

For $f=3$, Φ is determined by

$$(20)\qquad 6\Phi_3=S_1^3-3S_1S_2+2S_3$$

unless $p=2$ or 3. For $p=3$, we treat (20) as an algebraic identity and find that the terms in $x_R^2x_S$ and $x_Rx_Sx_T$ are multiples of 3; we thus obtain C_{RRS} and C_{RST} algebraically in terms of the characters. In

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*Or directly, by treating (17) as an algebraic identity, one half the coefficients of $x_Rx_S(R\neq S)$ are given by (19_1).

$$(21) \qquad S_4 - \Phi_1 S_3 + \Phi_2 S_2 - \Phi_3 S_1 = 0,$$

we select the terms x_R^4 and $x_R^3 x_S$, using (17), and obtain $C_{RRR}\chi(R)$ and $C_{RRR}\chi(S)$ as functions of the characters. But if every character vanishes, Φ involves only the cubes of the variables and is reducible modulo 3.

For $f = 3$, $p = 2$, Newton's identities give

$$(22) \quad S_3 + S_1^3 = \Phi_3 + \Phi_2 S_1, \quad \Phi_2(S_3 + S_1^3) = S_5 + S_1^2 S_3.$$

In Φ_2, C_{RS} is given by (19_1); to find C_{RR} we examine the coefficients of $x_R^i x_S^{5-i}$ $(i = 5, \cdots, 2)$ in (22_2). If in each the factor of C_{RR} vanishes then $S_3 + S_1^3 = 0$, identically. But Φ_3 would then be reducible by (22_1). Hence Φ_2 and then Φ_3 can be expressed in terms of the characters.

6. THEOREM. *In the field F_p any commutative group H of order $h = p^\pi q$ (q not divisible by the prime p) has exactly q distinct characters.*

If A is an element of period p^t of H, and we set $\chi(A) = W$, then in the field F_p we have

$$W^{p^t} = 1, \quad (W - 1)^{p^t} = 1, \quad W = 1.$$

Proceeding as in Weber's Algebra, II, Chapter 2, we conclude that H has at most $h/p^\pi = q$ different characters. But at least this number occur, since there exist in F_p primitive roots of $x^k = 1$ for k prime to p. In fact, $x^k = 1$ has no double root and hence k distinct roots in F_p.

7. THEOREM. *If C denotes the commutator group of H, the number of distinct linear factors in F_p of the group determinant of H is obtained by dividing the order of the quotient group $Q = H/C$ by the highest power of p dividing the order of Q.*

The proof follows from §6 and the argument in Frobenius's paper, *Sitzungsberichte*, 1896, pages 1347–1349. As there shown the linear factors occur to the same power. But the argument showing that this power is the first is not valid in F_p (cf. §8).

8. THEOREM. *If p^π is the highest power of p which divides the order of a group H, every irreducible factor, in the field F_p, of the regular group determinant of H enters to a power which is an exact multiple of p^π. For any character of degree f, $\chi(P) = f$ if P is an element of period a power of p.*

The proof is given in the July number of the *Transactions*.

9. The characters in the field F_p of a group H whose order

h is prime to p may be taken to be the algebraic characters given by Frobenius's theory. We shall consider examples in which h is divisible by p. If H is abelian or if h is a power of p, the characters follow from §§ 6, 8.

The relations obtained in § 4 do not in general completely determine the simple characters in F_p. Relation (IX) is equivalent to that obtained by replacing S and T by any elements conjugate to them or by interchanging S and T. Similarly for (VI) and (X); in the latter it suffices to set $T =$ identity. Not all the values of a simple character are zero (§ 5). If $\Phi = \Phi'\Phi''$ then $\chi(R) = \chi'(R) + \chi''(R)$ for every R.

10. Let H be the symmetric group on three letters. Let E, A, B be elements of period 1, 3, 2; and set $\chi(E) = f$, $\chi(A) = a$, $\chi(B) = b$. Then, by § 4,

$$f^2 + 2a^2 + 3b^2 = 6, \quad a^2 + 2af + 3b^2 = ca, \quad 4ab + 2fb = cb,$$

$$ca^2 = 3a + 3f, \quad cab = 6b, \quad cb^2 = 2f + 4a.$$

For modulus 2, the solutions (other than $f = a = b = 0$) are

(23) $f_1 = a_1 = b_1 = 1, \quad c_1 = 0; \quad f_2 = b_2 = 0, \quad a_2 = c_2 = 1.$

Hence by § 8, $|H| = \Phi_1^2\Phi_2^2$. Here Φ_2 is irreducible (end of § 9).

For modulus 3, the conditions reduce to $f = a$, $c = 0$. But by §§ 7, 8, $|H| = \Phi_1^3\Phi_1'^3$.

11. Let H be the alternating group on four letters. Let f, x, y, z be the values of a character when the arguments are non-conjugate elements of period 1, 2, 3, 3. Then, by § 4,

(24) $f^2 + 3x^2 + 8yz = 12, \quad 2fx + 2x^2 + 8yz = cx,$

(25) $2fy + 6xy + 4z^2 = cy, \quad 2fz + 6xz + 4y^2 = cz,$

(26) $4f + 8x = cx^2, \quad 12y = cxy, \quad 12z = cxz,$

(27) $3f + 9x = cyz, \quad 12z = cy^2, \quad 12y = cz^2.$

Let first the modulus p be 3. The solutions are

(28) $f=y=z=0, \ x=\pm 1, \ c=\mp 1; \quad f=x=y=z\neq 0, \ c=0.$

By § 5 or § 8, $f = 1$ for the second, $f = 3$ for the first. Further, the lower signs must be taken. Thus $|H| = \Phi_1^3\Phi_3^3$.

For $p = 2$, the relations reduce to $c = 0$, $x = f$. By §§ 7, χ, we have $|H| = \Phi_1^4\Phi_1'^4\Phi_1''^4$. It follows from (17) that $8^2(R) \equiv \chi(R^2)$, whence $x = f$, $y^2 = z$, $z^2 = y$.

12. Finally, let H be the alternating group on five letters. There are five sets of conjugate elements, of periods 1, 3, 2, 5, 5. Denote the corresponding values of $\chi(R)$ by f, A, B, C, D. By § 4,

$$(29) \qquad f^2 + 20A^2 + 15B^2 + 12C^2 + 12D^2 = 60,$$

$$(30) \qquad \begin{aligned} &2fA + 7A^2 + 12AB + 6AC + 6AD + 3B^2 \\ &\qquad + 6BC + 6BD + 3C^2 + 6CD + 3D^2 = cA, \end{aligned}$$

$$(31) \qquad \begin{aligned} &2fC + 5A^2 + 10AB + 10AC + 10AD + 5B^2 \\ &\qquad + 10BD + 5C^2 + 2CD + D^2 = cC, \end{aligned}$$

$$(32) \qquad \begin{aligned} &2(fB + 4A^2 + 4AB + 4AC + 4AD + 4BC \\ &\qquad + 4BD + B^2 + 4CD) = cB, \end{aligned}$$

$$(33) \qquad 3(f + 7A + 6B + 3C + 3D) = cA^2,$$

$$(34) \qquad 15(A + B + C + D) = cAC,$$

$$(35) \qquad 3(8A + 4B + 4C + 4D) = cAB,$$

$$(36) \qquad 20(A + B + D) = cBC, \quad 5(f + 5A + 5C + D) = cC^2,$$

$$(37) \qquad \begin{aligned} &4(f + 4A + 2B + 4C + 4D) = cB^2, \\ &\qquad 5(5A + 5B + C + D) = cCD, \end{aligned}$$

together with the relations derived from (31), (34) and (36) by interchanging C with D. Now Φ_f must equal, or be an irreducible factor in F_p, of one of the algebraically irreducible factors of the group determinant

$$|H| = \Phi_1 \Phi_3^3 \Phi_3'^3 \Phi_4^4 \Phi_5^5.$$

Let $p = 5$. If $c \neq 0$, we obtain

$$(38) \qquad f = C = D = 0, \quad A = \pm 1, \quad B = \mp 1, \quad c = \pm 3.$$

For $c = 0$, the conditions (29)–(37) reduce to

$$(39) \qquad f = C = D, \quad B = 3f + 3A, \quad c = 0.$$

In (38) $f = 5$, since $f < 10$ by § 5. The only algebraically irreducible factor of degree $\geqq 5$ has $f = 5$, $C = D = 0$, $A = -1$, $B = 1$. Hence the lower signs must hold in (38). If the resulting function Φ_5 were reducible, $\Phi_5 = \Pi \Phi_i$, the characters of the irreducible factors would satisfy (39) and $\sum A_i = A$, etc. As this is seen to be impossible, Φ_5 is irreducible in F_5. The remaining irreducible factors of $|H|$ satisfy (39) and have $f \leqq 4$. But the algebraically irreducible factor Φ_4 occurs only to the fourth power; hence by § 8 it is reducible in F_5. Now $\chi_4(R) \equiv \chi_1(R) + \chi_3(R)$. Hence $\Phi_4 = \Phi_1 \Phi_3$ in F_5. Finally, there is no factor Φ_2, since H cannot be represented as a binary linear group in F_5, the only binary transformation of period 2 being $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Hence the algebraic factor Φ_3 remains irreducible in F_5. Hence

$$(40) \qquad |H| = \Phi_1^5 \Phi_3^{10} \Phi_5^5 \text{ in } F_5.$$

Let next $p = 3$. If $c \neq 0$, we obtain

$$(41) \quad f = A = 0,\ c = B = \pm 1,\ C \text{ and } D \text{ roots of } y^2 \pm y - 1 = 0.$$

If $c = 0$, the conditions (29)–(37) reduce to

$$(42) \qquad f = A,\ D = C,\ A + B + C = 0,\ c = 0.$$

Proceeding as for $p = 5$, we find that the lower signs must hold in (41) and that the resulting functions Φ_3 and Φ_3' are irreducible in F_3; also that $\Phi_5 = \Phi_1 \Phi_4$. Thus

$$(43) \qquad |H| = \Phi_1^6 \Phi_3^3 \Phi_3'^8 \Phi_4^9 \text{ in } F_3.$$

Finally, let $p = 2$. If $c \neq 0$, we obtain

$$(44) \qquad f = B = 0,\ A = C = D = 1,\ c = 1.$$

If $c = 0$, conditions (29)–(37) reduce to

$$(45) \qquad f = B, \quad A + B + C + D = 0.$$

By the argument employed when $p = 5$, we find that the algebraically irreducible function Φ_4, whose characters are congruent to those in (44), remains irreducible in F_2. Since H is simply isomorphic with the group of all binary transformations of determinant unity in the $GF[2^2]$, there exist factors Φ_2 irreducible in F_2. Their characters are

$$(46) \quad f = B = 2, \quad A = 1, \quad C \text{ and } D \text{ roots of } y^2 + y + 1 = 0.$$

We find that in F_2,

$$\Phi_5 = \Phi_1 \Phi_2 \Phi_2', \quad \Phi_3 = \Phi_1 \Phi_2, \quad \Phi_3' = \Phi_1 \Phi_2'.$$

Hence

$$(47) \qquad |H| = \Phi_1^{12} \Phi_2^8 \Phi_2'^8 \Phi_4^4 \text{ in } F_2.$$

13. Frobenius's first method of introducing group characters (*Sitzungsberichte*, 1896, page 985) relates primarily to the factorization, into linear factors, of the special group determinant $|x_{PQ^{-1}}|$, in which $x_A = x_B$ if A and B are conjugate elements of the group H. This method is not suitable for the foundation of the modular theory; in fact, two distinct irreducible factors of the general group determinant G may correspond in the field F_p to the same linear factor of the special group determinant S. Thus if H is the symmetric group on three letters, and $p = 3$, $S = (x_E - x_A)^6$, while G has two distinct factors (§ 7 or § 10). It may be argued indirectly (cf. end of § 4) that the prime factors of Frobenius's fundamental determinant $|p_{\alpha\beta}|$ are all divisors of the order of H.

Frobenius's second method (l. c., page 1343) relates initially to the general group determinant G. With the exceptions noted in §§ 5, 7, the developments in his first three sections are valid in the field F_p. But his derivation in § 5 of the relations between the characters is not valid in F_p, when p divides the order h of the group. In fact, the power e to which any irreducible factor occurs in G is a multiple of p (§ 7), so that $\partial\Phi^{\bullet}/\partial x_R$ vanishes identically in F_p. His method, when applied to F_p, gives rise only to relations all of whose coefficients vanish.

Burnside's first treatment, *Proceedings London Mathematical Society*, volume 29 (1898), pages 207–224, 546–565, depends upon the special group determinant. Although his auxiliary theorems were established by purely rational processes by the writer in the *Transactions*, volume 3 (1902), page 285, it was indicated in the last paper, that the method is limited to fields whose modulus does not divide the order of the group.

Burnside's second treatment, *Acta Mathematica*, volume 28 (1904), page 369, and *Proceedings London Mathematical Society*, volume 1 (1901), page 117, depends upon the existence of an invariant Hermitian form, and hence is not applicable in the construction of a general theory of modular groups.

The exposition by Schur has the advantage that after the modifications noted in § 3, it yields important results in the modular theory.

THE UNIVERSITY OF CHICAGO,
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ON THE SHORTEST DISTANCE BETWEEN CONSECUTIVE STRAIGHT LINES.

BY MR. JOSEPH LIPKE.

CERTAIN well-known geometric results concerning space curves and surfaces have been obtained by a discussion of the shortest distance between consecutive positions of a straight line moving continuously in space. These results have been gained by a discussion (recapitulated in § 1) of the *numerator* of the expression for the shortest distance.* It is the purpose of this paper to complete the discussion by examining (§§ 2–3) the *denominator* of the distance expression, placing special emphasis upon the conditions that the distance be an infinitesimal of the *second* order, and upon a geometric interpretation of this case.

§ 1. *Brief Discussion of the Numerator.*

The equations of the straight line are

$$(1)\qquad \begin{aligned} x &= az + p, \\ y &= bz + q, \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &= a(t)z + p(t), \\ y &= b(t)z + q(t), \end{aligned}$$

where a, b, p, q are analytic functions of a single variable t. We define the consecutive line by the equations

$$(2)\qquad \begin{aligned} x &= a(t + dt)z + p(t + dt), \\ y &= b(t + dt)z + q(t + dt), \end{aligned}$$

or

$$x = \left(a + a'dt + a''\frac{dt^2}{2\,!} + \cdots\right)z + \left(p + p'dt + p''\frac{dt^2}{2\,!} + \cdots\right),$$

$$y = \left(b + b'dt + b''\frac{dt^2}{2\,!} + \cdots\right)z + \left(q + q'dt + q''\frac{dt^2}{2\,!} + \cdots\right),$$

where $a' = da/dt$, $a'' = d^2a/dt^2$, $\cdots$. The formula for the shortest distance between lines (1) and (2) is given by †

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* Koenigs: Géométrie réglée: *Annales de la Faculté des Sciences de Toulouse*, vol. 6, pp. 38–40, 61–63.

Joachimsthal: Anwendungen der Diff. und Int. Rechnung, etc., pp. 182–184.

Knoblauch: Einleitung in die allgemeine Theorie der krummen Flächen, pp. 104–106.

† Laurent: Traité d'analyse, vol. 2, pp. 298–303.

$$(3)\quad d=\frac{\begin{array}{r}\left(p'dt+p''\frac{dt^2}{2!}+\cdots\right)\left(b'dt+b''\frac{dt^2}{2!}+\cdots\right)\\ -\left(q'dt+q''\frac{dt^2}{2!}+\cdots\right)\left(a'dt+a''\frac{dt^2}{2!}+\cdots\right)\end{array}}{\sqrt{\begin{array}{r}\left\{\left[b'dt+b''\frac{dt^2}{2!}+\cdots\right]^2+\left[a'dt+a''\frac{dt^2}{2!}+\cdots\right]^2\right.\\ \left.+\left[a\left(b'dt+b''\frac{dt^2}{2!}+\cdots\right)-b\left(a'dt+a''\frac{dt^2}{2!}+\cdots\right)\right]^2\right\}\end{array}}}$$

and expanding, we get

$$(4)\quad d=\frac{\begin{array}{r}\begin{vmatrix}b' & a'\\ q' & p'\end{vmatrix}dt^2+\left\{\frac{1}{2!\,1!}\begin{vmatrix}b'' & a''\\ q' & p'\end{vmatrix}+\frac{1}{1!\,2!}\begin{vmatrix}b' & a'\\ q'' & p''\end{vmatrix}\right\}dt^3+\left\{\frac{1}{3!\,1!}\begin{vmatrix}b''' & a'''\\ q' & p'\end{vmatrix}\right.\\ \left.+\frac{1}{2!\,2!}\begin{vmatrix}b'' & a''\\ q'' & p''\end{vmatrix}+\frac{1}{1!\,3!}\begin{vmatrix}b' & a'\\ q''' & p'''\end{vmatrix}\right\}dt^4+\left\{\frac{1}{4!\,1!}\begin{vmatrix}b^{\text{iv}} & a^{\text{iv}}\\ q' & p'\end{vmatrix}+\frac{1}{3!\,2!}\begin{vmatrix}b''' & a'''\\ q'' & p''\end{vmatrix}\right.\\ \left.+\frac{1}{2!\,3!}\begin{vmatrix}b'' & a''\\ q''' & p'''\end{vmatrix}+\frac{1}{1!\,4!}\begin{vmatrix}b' & a'\\ q^{\text{iv}} & p^{\text{iv}}\end{vmatrix}\right\}dt^5+\cdots\end{array}}{\sqrt{\begin{array}{r}\left[\left\{b'^2+a'^2+\left(ab'-ba'\right)^2\right\}dt^2+\left\{\frac{2}{1!\,2!}b'b''+\frac{2}{1!\,2!}a'a''\right.\right.\\ \left.+\frac{2}{1!\,2!}\left(ab'-ba'\right)\left(ab''-ba''\right)\right\}dt^3+\left\{\frac{2}{1!\,3!}b'b'''\right.\\ +\frac{1}{2!\,2!}b''^2+\frac{2}{1!\,3!}a'a'''+\frac{1}{2!\,2!}a''^2+\frac{2}{1!\,3!}\left(ab'-ba'\right)\\ \left.\left.\times\left(ab'''-ba'''\right)+\frac{1}{2!\,2!}\left(ab''-ba''\right)^2\right\}dt^4+\cdots\right]\end{array}}}$$

or briefly,

$$(5)\quad d=\frac{N}{D}=\frac{[2]dt^2+[3]dt^3+[4]dt^4+[5]dt^5+\cdots}{\sqrt{\{[\bar{2}]dt^2+[\bar{3}]dt^3+[\bar{4}]dt^4+[\bar{5}]dt^5+\cdots\}}}.$$

It is at once evident that, in general, d is an infinitesimal of the first order; hence

THEOREM I. *The shortest distance between two consecutive generators of a skew surface, is, in general, an infinitesimal of the first order.*

If $[2]\equiv\begin{vmatrix}b' & a'\\ q' & p'\end{vmatrix}=0$, $[3]$, which is $\frac{1}{2}d\,[2]/dt$, also vanishes, but $[4]$ does not vanish; then d becomes an infinitesimal of the third order. Now the condition that line (1) moves tangent to the space curve

$$(6)\qquad x=a\psi+p,\quad y=b\psi+q,\quad z=\psi$$

through every point of which one and only one of the lines (1) passes, is that

$$\psi=-\frac{p'}{a'}=-\frac{q'}{b'}\text{ or }\begin{vmatrix}b' & a'\\ q' & p'\end{vmatrix}=0;$$

and curve (6) takes the form

$$(7)\qquad x=\frac{pa'-ap'}{a'},\quad y=\frac{qb'-bq'}{b'},\quad z=-\frac{p'}{a'},$$

where $b'p'=a'q'$.

Thus our line generates a developable surface. Hence

THEOREM II. *If the shortest distance between two consecutive positions of a moving line is an infinitesimal of the third order, the line will generate a developable surface, and conversely.*

Again, if $\left|\begin{smallmatrix} b' & a' \\ q' & p' \end{smallmatrix}\right| = 0$ and $\left|\begin{smallmatrix} b'' & a'' \\ q'' & p'' \end{smallmatrix}\right| = 0$ simultaneously, we have, integrating these, the two sets of solutions (*i*) $b = ca + c_1$, $q = cp + c_2$ and (*ii*) $p = ka + k_1$, $q = kb + k_2$ for either of which every $\left|\begin{smallmatrix} b^{(n)} & a^{(n)} \\ q^{(m)} & p^{(m)} \end{smallmatrix}\right|$ vanishes identically, hence N vanishes identically, and d becomes zero. Now, under conditions (*i*) line (1) becomes

$$x = az + p, \quad y = (ca + c_1)z + cp + c_2 \tag{8}$$

and curve (7) takes the form

$$x = \frac{pa' - ap'}{a'}, \quad y = \frac{c(pa' - ap') + c_2 a' - c_1 p'}{a'}, \quad z = -\frac{p'}{a'}, \tag{9}$$

a curve lying in the plane $y = cx + c_1 z + c_2$, hence the line moves tangent to a plane curve. Under conditions (*ii*) line (1) becomes

$$x = az + ka + k_1, \quad y = bz + kb + k_2 \tag{10}$$

and curve (7) takes the form

$$x = k_1, \quad y = k_2, \quad z = -k, \tag{11}$$

a point, and the line generates a cone. Hence

THEOREM III. *If the shortest distance between two consecutive positions of a moving line is identically zero, the line will move tangent to a plane curve or generate a cone, and conversely.*

We also note that $N = 0$ when $a' = 0, b' = 0$, or $a =$ const., $b =$ const., *i. e.*, our line moves parallel to itself and generates a cylinder; but this needs further discussion since D also vanishes here.

Finally, in any case, N is always an infinitesimal of even order.*

THEOREM IV. *For special lines of a surface (skew or developable), the shortest distance between two consecutive generators may be an infinitesimal of higher order than the third, but always of odd order.*

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* Zindler : Liniengeometrie mit Anwendungen : Zweiter Teil, p. 13.

§ 2. *Discussion of the Denominator.*

The denominator of the expression for d is

$$(12)\quad \begin{aligned} D &= \sqrt{\{[\bar{2}]dt^2 + [\bar{3}]dt^3 + [\bar{4}]dt^4 + \cdots\}} \\ &= \sqrt{\{[b'^2 + a'^2 + (ab' - ba')^2]dt^2 + [b'b''} \\ &\quad + a'a'' + (ab' - ba')(ab'' - ba'')]dt^3 + [\tfrac{1}{3}b'b''' \\ &\quad + \tfrac{1}{4}b''^2 + \tfrac{1}{3}a'a''' + \tfrac{1}{4}a''^2 + \tfrac{1}{3}(ab' - ba')(ab''' - ba''') \\ &\quad + \tfrac{1}{4}(ab'' - ba'')^2]dt^4 + \cdots\}. \end{aligned}$$

We have

$$[\bar{3}] = \tfrac{1}{4}\frac{d}{dt}[\bar{2}].$$

Hence

1) $[\bar{3}]$ will vanish identically whenever $[\bar{2}]$ vanishes identically, and

2) In general, D is an infinitesimal of the first order.

Let us find the solution of the differential equation

$$[\bar{2}] \equiv b'^2 + a'^2 + (ab' - ba_{,})^2 = 0.$$

The only real solutions of this equation are at once seen to be $b' = 0$, $a' = 0$, or $b = \text{const.}$, $a = \text{const.}$; these cause D to vanish identically. Hence

3) $b = \text{const.}$, $a = \text{const.}$ (the real solutions of $[\bar{2}] = 0$) cause D to vanish identically.

Now $[\bar{2}] = 0$ has imaginary solutions. To find these let us write the equation in the equivalent forms

$$\text{(i)}\qquad (a^2 + 1)\left(\frac{db}{dt}\right)^2 + (b^2 + 1)\left(\frac{da}{dt}\right)^2 - 2ab\frac{db}{dt}\frac{da}{dt} = 0,$$

or

$$\text{(ii)}\qquad b^2 - 2ab\frac{db}{da} + (a^2 + 1)\left(\frac{db}{da}\right)^2 + 1 = 0,$$

or

$$\text{(iii)}\qquad b = a\frac{db}{da} \pm i\sqrt{\left(\frac{db}{da}\right)^2 + 1}.$$

This differential equation is in Clairaut's form, and its integral is

$$(13)\qquad b = ca \pm i\sqrt{c^2 + 1}. \quad (c = \text{constant.})$$

This gives rise to the system of equations

$$b' = ca', \quad b'' = ca'', \quad b''' = ca''', \cdots, \quad b^{(n)} = ca^{(n)}, \cdots$$

and it is easily seen from (3) that D vanishes identically. Hence

4) The general solution $b = ca \pm i\sqrt{c^2+1}$ of the differential equation $[\bar{2}] = 0$ makes D vanish identically.

Now the differential equation $[\bar{2}] = 0$ is one of second degree and therefore has a singular solution. The latter is most readily found by solving the form (*ii*) as a quadratic equation in db/da, and setting the discriminant equal to zero. The singular solution is found to be

$$a^2 + b^2 + 1 = 0. \tag{14}$$

This solution (14) causes $[\bar{2}]$ and $[\bar{3}]$ to vanish and, as can easily be found with a little calculation, makes $[\bar{4}] = -b'^4/4a^4$, *i. e.*, does not cause $[\bar{4}]$ to vanish. Hence

5) The singular solution $a^2 + b^2 + 1 = 0$ of the differential equation $[\bar{2}] = 0$ reduces D to an infinitesimal of the second order.

Again if $[\bar{4}]$ vanishes in addition to the vanishing of $[\bar{2}]$ through its singular solution, we must have, since $[\bar{4}] = -b'^4/4a^4 = 0$, $b' = 0$, or $b =$ const., which taken in conjunction with $a^2 + b^2 + 1 = 0$, gives also $a =$ const.; but $a =$ const., $b =$ const., cause D to vanish identically. Hence

6) The singular solution $a^2 + b^2 + 1 = 0$ of $[\bar{2}] = 0$ taken simultaneously with $[\bar{4}] = 0$ causes D to vanish identically.

Thus, we finally have

7) D is in general an infinitesimal of the first order; if $a^2 + b^2 + 1 = 0$, D is an infinitesimal of the second order; if $b = ca \pm i\sqrt{c^2+1}$, or if $b =$ const., a const., D is identically zero.

§ 3. *Geometric Interpretations.*

We have

$$d = \frac{N}{D} = \frac{[2]\,dt^2 + [3]\,dt^3 + [4]\,dt^4 + \cdots}{\sqrt{\{[\bar{2}]\,dt^2 + [\bar{3}]\,dt^3 + [\bar{4}]\,dt^4 + \cdots\}}}. \tag{15}$$

If $[\bar{2}] = 0$, through its general solution $b = ca \pm i\sqrt{c^2+1}$

and also $[2] \equiv \left|\begin{smallmatrix} b' & a' \\ q' & p' \end{smallmatrix}\right| = 0$, then $q'/p' = b'/a' = c$, or $q' = cp'$, and $q = cp + c_2$; but then both D and N vanish identically, and d takes the indeterminate form $0/0$. But this indeterminacy is easily resolved, for the conditions $b = ca \pm i\sqrt{c^2+1}$, $q = cp + c_2$, are only a special case of $b = ca + c_1$, $q = cp + c_2$, the conditions that the line move tangent to a plane curve. Thus d is zero. The curve (6) becomes

$$(16) \quad x = \frac{pa' - ap'}{a'},\ y = \frac{a(pa' - ap') + c_2 a' \mp i\sqrt{c^2+1}\,p'}{a'},\ z = -\frac{p'}{a'},$$

an imaginary curve lying in the imaginary plane

$$y = cx \pm i\sqrt{c^2+1}\,z + c_2.$$

THEOREM VI. *If a continuously moving straight line* $x = az + p$, $y = bz + q$ *obeys the conditions* $b = ca \pm i\sqrt{c^2+1}$, $q = cp + c_2$, *it will move tangent to an imaginary curve in an imaginary plane, the distance between two consecutive tangent lines being zero.*

Again, if $[\bar{2}]$ vanishes through its singular solution $a^2 + b^2 + 1 = 0$, then by 7) D is an infinitesimal of the second order, and if $[2]$ does not vanish, N is also an infinitesimal of the second order, and thus d is finite. Now the relation $a^2 + b^2 + 1 = 0$ expresses that the sum of the squares of three quantities proportional to the direction cosines of our line, is equal to zero, which property is the distinguishing characteristic of a minimal straight line of space. Hence

THEOREM VII. *The shortest distance between two consecutive generators of a skew surface generated by a continuously moving minimal straight line, is finite.*

If in addition to the vanishing of $[\bar{2}]$ through its singular solution $a^2 + b^2 + 1 = 0$, $[2] = \left|\begin{smallmatrix} b' & a' \\ q' & p' \end{smallmatrix}\right|$ also vanishes, N is an infinitesimal of the fourth order, for we have

$$a^2 + b^2 + 1 = 0 \quad \text{and} \quad \begin{vmatrix} b' & a' \\ q' & p' \end{vmatrix} = 0$$

or

$$aa' + bb' = 0 \quad \text{and} \quad \frac{b'}{a'} = \frac{q'}{p'} = -\frac{a}{b}.$$

$$\therefore q' = -\frac{a}{b}p' \quad \text{or} \quad q = -\int \frac{a}{b}p'dt\,.$$

$$\therefore b = \pm i\sqrt{a^2+1} \quad \text{and} \quad q = \pm i \int \frac{a}{\sqrt{a^2+1}} p'dt.$$

With these conditions we can easily calculate

$$\begin{vmatrix} b'' & a'' \\ q'' & p'' \end{vmatrix} = \frac{a'}{b^3}(a''p' - a'p'') \neq 0,$$

hence [4] does not vanish, *i. e.*, N is an infinitesimal of the fourth order. Thus d is an infinitesimal of the second order. The line moves tangent to a minimal curve in space. The equations of the line are

$$(17) \qquad x = az + p, \quad y = \pm i\sqrt{a^2+1}\, z \pm i \int \frac{a}{\sqrt{a^2+1}} p'dt,$$

and those of the minimal curve to which the line moves tangent are, from (6),

$$(18) \qquad x = \frac{pa' - ap'}{a'}, \qquad z = -\frac{p'}{a'},$$

$$y = \mp i \frac{\sqrt{a^2+1}\,p'}{a'} \pm i \int \frac{a}{\sqrt{a^2+1}} p'dt.$$

Hence

THEOREM VIII. *The shortest distance between any two consecutive generators of the tangential surface to a minimal curve in space, i. e., of a minimal developable, is an infinitesimal of the second order; and conversely, if a line moves so that the shortest distance between any two consecutive positions is an infinitesimal of the second order, it will generate a minimal developable.*

The converse as stated in Theorem VIII is easily deduced from the above discussion, for d is an infinitesimal of second order only if $[2] = 0$, $[4] \neq 0$, $[\bar{2}] = 0$ (through its singular solution) and $[\bar{4}] \neq 0$.

From equations (18) we have

$$x' = \frac{a(p'a'' - a'p'')}{a'^2}, \quad y' = \frac{\pm i\sqrt{a^2+1}(p'a'' - a'p'')}{a'^2},$$

$$z' = \frac{p'a'' - a'p''}{a'^2}$$

and hence, the equations of the minimal curve may be written

$$(19)\quad x=\int\frac{a(p'a''-a'p'')}{a'^2}dt,\quad y=\pm i\int\frac{\sqrt{a^2+1}(p'a''-a'p'')}{a'^2}dt,$$

$$z=\int\frac{p'a''-a'p''}{a'^2}dt.$$

Now set $a=(1-\tau^2)/2\tau$, where τ is an arbitrary function of t,

$$\therefore \pm i\sqrt{a^2+1}=\frac{i(1+\tau^2)}{2\tau},$$

and set $(p'a''-a'p'')/a'^2dt=\tau F(\tau)d\tau$, where $F(\tau)$ is an arbitrary function of τ. Then equations (19) become

$$(20)\quad x=\frac{1}{2}\int(1-\tau^2)F(\tau)d\tau,\quad y=\frac{i}{2}\int(1+\tau^2)F(\tau)d\tau,$$

$$z=\int\tau F(\tau)d\tau,$$

the well-known equations of the minimal curve.

Using equations (20), we have

$$[2]=0,\ [3]=0,\ [\bar{2}]=0,\ [3]=0,\ [4]=\frac{-iF(\tau)}{12\tau^2},\ [\bar{4}]=-\frac{1}{4\tau^4}$$

and

$$(21)\quad d=\frac{\dfrac{-iF(\tau)}{12\tau^2}d\tau^4+\cdots}{\sqrt{\left\{-\dfrac{1}{4\tau^4}d\tau^4+\cdots\right\}}}=-\frac{1}{6}F(\tau)d\tau^2+\cdots$$

Using equations (19), we have $[4]=-a'/12b^3\times(p'a''-a'p'')$, $[\bar{4}]=-b'^4/4a^4$ and

$$(22)\quad d=\frac{i}{6}\frac{p'a''-a'p''}{ba'}dt^2+\cdots=\frac{p'a''-a'p''}{6a'\sqrt{a^2+1}}dt^2+\cdots.$$

Finally we may have $[\bar{2}]=0$ through its singular solution $a^2+b^2+1=0$ and $[\bar{4}]=0$, *i. e.*, $a=\text{const.}$, $b=\text{const.}$ Here both N and D are zero, and hence d has the indeterminate form 0/0. Since $\psi=-p'/a'=-p'/0=\infty$, the curve (6) becomes $x=\infty, y=\infty, z=\infty$, *i. e.*, our lines all pass through the same point at infinity; but for the purpose in hand we cannot consider the cylinder as a special case of the cone, *i. e.*, a cone whose vertex is at infinity, for when $a = \text{const.}$, $b = \text{const.}$, the conditions that the line generates a cone, viz., $p = ka + k_1$, $q = kb + k_2$ become $p = \text{const.}$, $q = \text{const.}$ and the two consecutive generators actually coincide. Thus, we cannot say, that the shortest distance between two consecutive generators of a cylinder is zero, *i. e.*, that the two generators actually intersect. There is no shortest distance between two such lines; they are everywhere equally distant. Hence, to find the distance between two consecutive parallel lines, we shall have to use the formula for the distance of a point from a line. It is easily seen that, in general, this distance is an infinitesimal of the first order; it is zero only if the two consecutive lines coincide; it is infinite when $a^2 + b^2 + 1 = 0$. Hence

THEOREM IX. *The distance between two consecutive generators of a cylinder is, in general, an infinitesimal of the first order; if the generator is a minimal straight line, the distance is infinite.*

COLUMBIA UNIVERSITY.

NOTE ON THE COMMUTATOR OF TWO OPERATORS.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, April 27, 1907.)

THERE is a confusing lack of uniformity with respect to the use of the term commutator. The present note aims to exhibit this fact and to point out some of its sources in the hope that these data may tend towards greater uniformity in the use of this term and also make its various meanings less confusing to the reader.

The operation now known as the commutator of two operators was used for a long time in the development of group theory before it received a special name. It is frequently employed, in various forms, in Jordan's Traité des substitutions, and its elegant application in the study of direct products was recognized by Hölder* and others. The first paper which deals with the

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*Holder, *Math. Annalen*, vol. 34 (1889), p. 35. It should be noted that the reference 91) in Encyklopädie der mathematischen Wissenschaften, vol. 1, p. 219, should be to this article instead of to the later one in vol. 43.

important properties of the subgroups formed by
tors of every pair of operators in a finite group
Quarterly Journal of Mathematics in 1896.* In
commutator of s and t is represented by $sts^{-1}t^{-1}$,
name is assigned to the operation. Its form wa
the form of the alternant (Klammerausdruck) i
continuous groups.

Towards the close of 1896 Frobenius reproduc
fundamental theorems relating to the commutat
(giving due credit both to the earlier publication
kind's unpublished work along the same line)
operation $s^{-1}t^{-1}st$ the commutator of t and s, fo
kind.

Shortly after this Dedekind published an
Mathematische Annalen‡ in which he gave some
properties of the commutator subgroup and def
the commutator of t and s just as Frobenius had
a year later $s^{-1}t^{-1}st$ was defined § as the commuta
instead of the commutator of t and s as had bee
articles just cited, and it was observed that the
s and t in $s^{-1}t^{-1}st$ gives its inverse, thus leading
mutators of s and t if the order of these opera
served.

In Weber's Lehrbuch der algebra, volume
133, the second definition of commutator given ab
so that Weber's commutator is the inverse of
Frobenius and Dedekind. Some writers say th
two operators $s^{-1}t^{-1}st$ and $t^{-1}s^{-1}ts$ are commutat
thus giving a double meaning to this term.
appears implicitly in the last article cited above
by Easton in his Constructive development of
1902, page 57 and also by De Séguier in his Gro
1904, page 8. A disadvantage of this definiti
expression "commutator of s and t" may mean
two operators.

If the elements of the commutator $s^{-1}t^{-1}st$ ar
every possible manner there result eight operato
be distinct and differ from the identity. Each o

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* Vol. 28, p. 266.
† Frobenius, *Berliner Sitzungsberichte*, 1896, p. 1348.
‡ Dedekind, *Math. Annalen*, vol. 48 (1897), p. 553.
§ BULLETIN, vol. 4 (1898), p. 135.

property that it is explicitly the transform of an operator multiplied by its inverse, and hence all of them have been called commutators.* Four of them are conjugate under $\{s, t\}$, while the others are the inverses of these four. Hence they have the same order and all of them occur in the commutator subgroup of the group generated by s and t. In fact they generate this subgroup. As long as we are concerned only with the properties of the commutator subgroup, it does not matter which of the eight forms considered above is regarded as the commutator of s and t. In fact, all of them may be regarded as commutators of these elements in this connection. In view of the fact that the commutator subgroup, has played the principal rôle in the discussions in which the term commutator has been employed, it is not surprising that there should have been some laxity in the definition of the term.

While the eight commutators considered above play the same rôle with respect to the commutator subgroup, they have quite different properties if considered as factors of s and t. For instance, $s^{-1}t^{-1}st$ is the factor which multiplied on the right into s $(s \cdot s^{-1}t^{-1}st)$ gives the transform of s with respect to t, but it has no such property as regards t. In fact, the product of this commutator into t will generally have an order which differs from the order of t and hence there is, in general, no operator which transforms t into itself multiplied by this commutator. From this it follows that $s^{-1}t^{-1}st$ has entirely different properties with respect to the two operators s, t and the question arises whether it would not be desirable to select for the commutator of s, t an operator whose properties with respect to s and t are more nearly alike.

Two of the eight commutators ($s^{-1}t^{-1}st$, $t^{-1}sts^{-1}$, $sts^{-1}t^{-1}$, $ts^{-1}t^{-1}s$, $t^{-1}s^{-1}ts$, $st^{-1}s^{-1}t$, $tst^{-1}s^{-1}$, $s^{-1}tst^{-1}$) which are obtained by permuting the elements of $s^{-1}t^{-1}st$ are such that the products obtained by multiplying them into s or t will be of the same order as s or t respectively. Two others will not change the order of s, but will generally change the order of t, if used as factors with s or t, while two others will not change the order of t but will change the order of s. The remaining two will, in general, change the order of both t and s if they are multiplied into these operators respectively. From this standpoint it does not appear desirable to call either $ts^{-1}t^{-1}s$ or $st^{-1}s^{-1}t$ a commutator of s and t. That is, if we regard the commutator as a factor which must be multiplied into an operator in order to obtain its conjugate (and this is a very useful concept) neither of the two operators $ts^{-1}t^{-1}s$, $st^{-1}s^{-1}t$ is a commutator of s or t since we cannot generally transform s or t into themselves multiplied by one of these two operators. From this point of view $s^{-1}t^{-1}st$ and $tst^{-1}s^{-1}$ are commutators of s but not of t, $t^{-1}s^{-1}ts$ and $sts^{-1}t^{-1}$ are commutators of t but not of s, while $t^{-1}sts^{-1}$ and $s^{-1}tst^{-1}$ are commutators of both s and t.

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* BULLETIN, vol. 5 (1899), p. 239. The general definition of a commutator is "the product of the transform of an operator and its inverse." On this point all authors agree. The disagreements relate to the definition of a commutator by means of its elements. When we speak of the commutators of a group it is assumed that the elements of the commutators are also found in the group and hence it may happen that only a small number of its operators are commutators. Since every operator of a group may be represented by a positive substitution and all positive substitutions are commutators, it follows that every possible operator is a commutator of some elements.

From the preceding paragraph it follows that for some important uses of the commutator it would appear desirable to call $t^{-1}sts^{-1}$ the commutator of s and t, and $s^{-1}tst^{-1}$ the commutator of t and s, while $s^{-1}t^{-1}st$ and $t^{-1}s^{-1}ts$ might be called the commutators of s with respect to t and of t with respect to s respectively. Although the main objects of this note are to call attention to the history and the nature of the commutator, yet the reasons given above seem to demand a new definition of the term and we shall hereafter call $t^{-1}sts^{-1}$ *the commutator of s and t.* If this is done the eight commutators mentioned at the beginning of the preceding paragraph are respectively the commutators of t^{-1} and s, s and t, t and s^{-1}, s^{-1} and t^{-1}, s^{-1} and t, t^{-1} and s^{-1}, s and t^{-1}, t and s. These eight commutators are conjugate under the octic group.

It is well known that every simple isomorphism of a group G with itself may be obtained by transforming all of its operators by some operator in its holomorph. In such an isomorphism each operator corresponds to itself multiplied by some operator of G. If G involves s and t, and if we transform G by t, the operator which corresponds to s may be obtained by multiplying s on the left by the commutator of s and t. By multiplying t on the right by the same commutator we obtain its transform with respect to s^{-1}. As in many instances (such as finding the operators of G which transform a function into one of its conjugates) it is desirable to employ left hand multiplication, the advantages of the last definition of commutator become apparent. In conclusion it may be said that it seems very desirable that the term commutator should be completely defined by giving its elements in order. That is, the expression '*commutator of s and t*' should not have a double meaning. For the most important applications which have been made of commutators any one of the given definitions seems just as good as any other, but there are applications in which the last definition seems to be the most convenient. It may be added that the definition of commutator in the Encyklopädie der Mathematischen Wissenschaften, Volume I 1, page 210, is rendered meaningless by typographical errors.

A THEOREM IN THE THEORY OF NUMBERS.

BY PROFESSOR D. N. LEHMER.

(Read before the San Francisco Section of the American Mathematical Society, December 19, 1903.)

LAGRANGE has shown that if the indeterminate equation $x^2 - Ry^2 = \pm D$ is resolvable in integers, D being less than $\sqrt{R}$, and x and y being relative primes, then D is a denominator of a complete quotient in the expansion of $\sqrt{R}$ in a continued fraction. (For a proof of this theorem, see Chrystal's Algebra II, page 451.) Making use of this result, we may prove the following interesting theorem, which is sometimes very effective in finding the factors of large numbers.

If R is the product of two factors which differ by less than $2\sqrt[4]{R}$, these two factors may be found directly from the expansion of $\sqrt{R}$ in a continued fraction.

Let the two factors be p and q, so that $R = pq$. Then $R = [\frac{1}{2}(p+q)]^2 - [\frac{1}{2}(p-q)]^2$, and the equation $x^2 - Ry^2 = [\frac{1}{2}(p-q)]^2$ is resolvable in integers. If now $[\frac{1}{2}(p-q)]^2$ is less than $\sqrt{R}$, then by the theorem quoted above, there will be a denominator of a complete quotient in the expansion of $\sqrt{R}$ equal to $[\frac{1}{2}(p-q)]^2$. Since $[\frac{1}{2}(p-q)]^2 < \sqrt{R}$, then $p - q < 2\sqrt[4]{R}$. Moreover the values of the indeterminates in the equation $x^2 - Ry^2 = \pm D$, are furnished by the numerator and denominator of the convergent which immediately precedes the complete quotient having D for a denominator. Hence it follows that the expansion of $\sqrt{R}$ need not be carried farther than is sufficient to make the numerator of the convergent as great as R. The method may be applied to κR, and we have the following theorem:

THEOREM: *If no denominator of odd rank, after the first, in the complete quotients obtained by expanding* $\sqrt{\kappa R}$ *in a continued fraction, turns out to be a perfect square, the expansion being carried out until the numerator of the last convergent is greater than* κR, *then the factors of* κR *differ by more than* $2\sqrt[4]{\kappa R}$.

As an example of the application of this theorem to the discovery of prime factors, take Jevons's[*] number $8616460799 = R$. No perfect square appears in the denominators of the complete quotients obtained in expanding $\sqrt{R}$, whence one infers that the factors differ by more than $608 = 2\sqrt[4]{R}$. Similar failure attends the expansion of $\sqrt{2R}$, $\sqrt{6R}$, and $\sqrt{30R}$. On expanding $\sqrt{210R}$, however, the third denominator is found to be the square $11881 = 109^2$. The numerator of the second convergent is 2690321; we know then that the numbers 2690321 ± 109 contain the desired factors of the given number. The factors are 89681 and 96079.

The most advantageous value of κ to take is the product of the smallest distinct primes. Thus if $\kappa = 30$ and the factors of R are p and q, then the factors will be discovered if $p - 30q$, $2p - 15q$, $3p - 10q$, or $5p - 6q$ are less than $2\sqrt[4]{30R}$.

BERKELEY, CAL.

PROJECTIONS OF THE GLOBE APPROPRIATE FOR LABORATORY METHODS OF STUDYING THE GENERAL CIRCULATION OF THE ATMOSPHERE.

BY PROFESSOR CLEVELAND ABBE.

THE general circulation of the atmosphere is controlled by the general distribution of land and water, and by the insolation, with its resultant temperature, evaporation and clouds. In the analytic treatment of this problem, beginning with D'Alembert, Ferrel, and Erman, as well as in the more elegant works of Helmholtz, Oberbeck and Margules, it has always been considered necessary to simplify the problem by assuming a uniform surface and uniform coefficient of resistance for the whole globe, as also a uniform condition of dry air without the complexity produced by the presence of clouds. In this shape the problem is already too difficult for pure analysis, and writers have generally achieved the special solutions only by introducing numerical values based upon observations of temperature, pressure, and winds that represent existing conditions upon the earth's surface. It is likely that the profound meteorological problem, namely the *deduction* of the actual winds that necessarily result from the laws of mechanics, as applied to the earth, will for a long time be too difficult for pure analysis; but the solution of this problem is greatly to be desired, and on many occasions I have stated my belief that it may be arrived at in an experimental way provided we have a laboratory properly equipped for meteorological investigations. We cannot reproduce atmospheric conditions surrounding a small globe in the laboratory, because in the free atmosphere the force of gravity is directed normal to the earth's surface, whereas in the laboratory it would be inclined to a globe at every possible angle. We must, therefore, represent any portion of the globe, with its atmosphere, by a horizontal plane surface covered with some heavy gas or liquid. The most natural thought is to adopt polar projections of the northern and southern hemispheres respectively, bounded at or beyond their equators by flexible rims that shall represent the assumed irregular variable equatorial boundary between the northern and southern circulations. This will do for those months of the year during which we may assume that the northern and southern systems do not intrude upon each other to any great extent; but of course this will fail entirely when the great Asiatic monsoon is in progress and for that season a very different laboratory experiment must be devised. We therefore for the present confine ourselves to the months October to March. Our projections of the two hemispheres, being covered with a thin layer of carbonic acid gas (or for that matter any liquid such as alcohol or water can be used as well), are to be set in steady rotation, and are to be warmed from below in such a way as approximately to imitate the actual isotherms of the lower atmosphere for any moment or day. Not only the general distribution of temperature in latitude, but the special areas of high and low temperatures, can easily be imitated by electrical resistance coils. In order that the thin layer which imitates the atmosphere shall be free from the action of the surrounding air of the laboratory our rotating planospheres are to be covered by plates of glass rotating with them, so that the upper layer of air in the model shall, like the upper layer in the atmosphere, rotate with the surrounding ether.

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[*] Jevons, Principles of science, p. 123, "Can the reader say what two numbers multiplied together will produce the number 8616460799? I think it unlikely that anyone but myself will ever know." I think that the number has been resolved before, but I do not know by whom.

We have thus imagined a construction that has a certain similarity but many points of dissimilarity to the atmosphere of the earth. Our problem is to reduce these latter to a minimum, and then so to interpret the observed motions in our model that we may infer something as to the motions in the earth's atmosphere. To do this our first step must be to consider the ideas laid down by Helmholtz as to the "mechanical similarity" of the atmosphere of the earth and the model atmosphere. Various theorems relative to this similarity were first published by Helmholtz in 1873, "On a theorem relative to movements that are geometrically similar in fluid bodies," and these ideas were further applied by him in later memoirs on atmospheric motions, in 1888 and 1889. He shows how the movements in a small model may be made geometrically and mechanically similar to those of the earth's atmosphere. The equations of motion for the model atmosphere become analogous to those of the free atmosphere by introducing certain conditions represented by three constant factors, *i. e.*, I, the viscosity of the model or liquid is to be a certain multiple q of the viscosity of the atmosphere; II, the density in the model is to be a certain multiple r of the density of the atmosphere; III, the velocities along three coordinate axes in the model are to be equal to those in the atmosphere multiplied by a certain constant factor n. Under these conditions the pressures in the model will be n^2r times the pressures in the atmosphere. The linear distances in the model will be q/n times the linear distances in the atmosphere, and the time argument in the model will be q/n^2 times the time element in the free air. It is assumed that the relative movements are not so violent as to depart greatly from what is known as laminar or lamellar motion, and that the viscosity may be combined with the resistances due to convection and to the formation of discontinuous motions. Numerous applications of these principles are given by Helmholtz, and some further developments have lately been added by Lord Rayleigh.

But in applying these ideas to our two polar projections we stumble at once upon a great difficulty, namely that the maps are not true representations of the spherical surface of the earth. Every projection on a plane must distort the geometric relations between the points upon the original spherical surface, and students must always choose a style of projection that suits the problem they have on hand. In the present case (1) we should like to preserve the equality of surface areas, since that is proportional to the mass of the air above it; (2) we should like to preserve equality of distances, since we have to measure and compare velocities; but, above all, from a mechanical point of view, (3) we should like to preserve the equality of the moments of inertia, that is to say the moment of inertia of a unit mass relative to the earth's axis should be a constant fraction of the moment of inertia of the corresponding unit mass in the model with reference to the pole about which it is rotating. These three desiderata would respectively necessitate the use of (1) an equal surface development; (2) a polyconic development; and (3) an orthographic projection. There are still other conditions with projections that correspond thereto. That which best satisfies all desiderata with the minimum sum total of errors seems to be Airy's "projection by balance of errors," published by him in 1861 as suggested by the projection used by Sir Henry James, in which latter the point of view was at the distance from the surface of the sphere equal to the radius. James's idea was to obtain a projection that should include a very large part of the surface of the earth at one view; and the map that he produced did in fact include a radius of 120° 0′; this was a projection, properly so called. Airy, however, solved the problem for a development, properly so called, which should respond to the condition that the square of the distortion in distance plus the square of the distortion in area should be a minimum when one considers the sum total of the whole surface. This minimum would necessarily depend on the total area included in the map. A few months later R. A. Clarke showed that there was a slight error in Airy's computations, and then worked out a slight modification of the problem. Clarke deduced a projection, properly so called, allowing both the visual point and the plane of projection to be adjustable, so as to satisfy the same condition as that adopted by Airy in his development. The formulas and computations by Clarke's projection are far more troublesome than for Airy's development, especially when the latter are arranged in the way that I have given. The only advantage that Clarke's projection can have is that it can be constructed graphically without the use of numerical tables or computations; but this advantage disappears as soon as we attempt to construct a large map.

The combination of Airy's development (or some projections still more appropriate) with Helmholtz's method of mechanical similarity should enable us to interpret our laboratory experiments intelligently, so that from these we may construct a close approximation to the general circulation of the atmosphere.

I consider it extremely desirable that these experiments should be made on a large scale, with due regard to all numerical, statistical and mechanical details in some laboratory where the study of meteorology is prosecuted as a branch of mathematical physics.

SHORTER NOTICES.

Breve Storia della Matematica dai tempi antichi al medio evo. By GAETANO FAZZARI. Milano, R. Sandron, 1907. 268 pp. Price, 4 lire.

IT is rather strange that Italy, the country that produced the most learned bibliophile in the domain of mathematics, Prince Boncompagni, and that furnished to France another well-known collector of early works, the historian Libri, should have published so little relating to the general development of the science. It is true that Favaro and Loria have contributed very acceptably to the history of certain periods or topics, and that Riccardi's bibliography of the early mathematical works of his own country will always be a standard book of reference, but in spite of all the encouragement of men like these, and all the patriotism that would lead an Italian to write the story of a science that so largely developed on his native soil, such a work as a worthy general history of mathematics does not exist in the language of Italy. It is for this reason that such an attempt as Professor Fazzari's should be particularly welcome, the more so as it was written on the island in which Archimedes spent most of his life, which Pythagoras visited, and to which Maurolycus brought no small amount of glory in the period of the Renaissance.

The aim of Professor Fazzari has been to write a work of about the character of Ball's and Cajori's popular histories, one that tells the story of mathematics in such a simple manner as to appeal to the younger student rather than to the mature scholar, leading him to higher fields in the science by an interesting recital of its progress. He has not, therefore, felt it necessary to pay attention to the original sources of information, but has gathered his material from a few of the standard writers, setting it forth in a pleasing conversational style. The general range of the subjects may be seen from the table of contents: Chapter I, Decimal numeration; Chapter II, The Egyptians; Chapter III, The Babylonians; Chapter IV, Logistica among the Greeks; Chapter V, The pre-euclidean period, including Thales and the Ionic school, the mathematics of the fifth century B. C., and the Academy; Chapter VI, The golden period of Greek geometry; Chapter VII, The Greek mathematics of the second century B. C.; Chapter VIII, The period of decadence; Chapter IX, The Romans; Chapter X, The Indians; Chapter XI, The Arabs; Chapter XII, The Byzantine school; Chapter XIII, The middle ages, from the seventh to the fifteenth centuries inclusive. It will thus be seen that the general field covered is about what would be expected, and that the result must be helpful to students and teachers in Italy if the work has been tolerably well done.

It is a little difficult to answer the question which the preceding sentence suggests. What is meant by "tolerable" in such a work? Where does the intolerable begin, and what is the norm of comparison? The book is better than Hoefer's, as in the natural order of things it could hardly help being; it is not so good as Ball's, and indeed it would be difficult to improve upon the popular style of that writer; it is not in the same class with such scholarly productions as those of Montucla, Libri, and Cantor, and of course this was not to be expected. It is but just to say that it is pleasantly written and that it covers the leading topics down to the opening of the sixteenth century; but it cannot be said that it shows a very wide range of reading, or that Professor Fazzari has produced an entirely reliable work.

With respect to the authorities consulted, the care shown in securing data, and the weighing of evidence, a few illustrations will suffice. In speaking of the Roman numerals, but two theories are given to explain the symbols, both rather antiquated, and no evidence is shown of any knowledge of the important investigations of Friedlein, Hoüel, Zangemeister, Mommsen, and R. Bombelli. As a consequence, the treatment of the subject has no real value, if, indeed, it can be called fairly reliable. In the rather fanciful etymologies of the lower numerals, the Humboldt-Curtius theory as to *quinque*, now generally agreed to be untenable, is given as the latest pronouncement in the case. The illustration (Fig. 3) from the Ahmes Papyrus is not only incorrectly drawn (having the hieroglyphic for 37 instead of 33), but it is not from the papyrus at all, being merely Eisenlohr's hieroglyphic rendering of the hieratic original. The positive statement that Heron of Alexandria was living in the period 120–100 B. C. shows that the author has not consulted Schmidt's 1899 edition of the Opera, and the passing of the name of Menelaus with no reference to the anharmonic ratio shows that he has not consulted the Abhandlungen for 1902. The treatment of algebraic symbolism (page 127) leaves an entirely wrong impression as to the early use of the common signs of operation and relation, and it seems doubtful if the author himself is entirely clear in this matter. On the question of the origin of the Hindu-Arabic numerals evidently no attempt has been made to trace these forms back to any of the pre-Christian cave inscriptions, and the statement on page 177 indicates that the author is not aware of the existence of these sources. That the numerals came primarily from the Sanskrit alphabet, as stated on page 179, is now an abandoned theory.

These criticisms fairly represent the lack of scholarly effort in the preparation of the work. Similar ones will suggest themselves to the reader at intervals so frequent as to lead him to regret that the author did not put more time upon the preparation of his manuscript and revise the proof more carefully.

As to the description of the Babylonian numerals, it was not to be expected that the recent Hilprecht discoveries would be known to Professor Fazzari, but these make his treatment of the matter seem very elementary.*

As a piece of bookmaking the work leaves much to be desired, and the absence of an index in such a publication is indefensible. Of the minor errors, chiefly in the proof-reading, the following are types: Händbuch for Handbuch (p. 19), the letter sigma for stigma (p. 35), und for and (p. 56), 1491 for 1494 (Paciuolo, p. 127), Ceilan for Ceulen (p. 171), Geometria for La géométrie as the title of the first edition of Descartes (page 184), and the double spelling of Bhaskara (pages 153, 173) and Paciuolo (pages 127, 259). The form "Mohammed ben Musa Al Hovarezmi" is probably the least satisfactory of any for the name of the great Arab mathematician, particularly as it is followed by the statement that he was a native of Chwarizm, and as the form "Alhowarizmi" appears on page 207. It is unfortunate that we have as yet no generally accepted norm for such transliterations, but there is no good authority for such a mixture of languages as this. A similar criticism might justly be passed upon most of the other oriental names in the work, particularly Al Fahri (page 194), Al Karhi and Alkarhi (pages 194, 195), and Alhayyami (page 199).

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*See the BULLETIN, vol. 13, p. 392.

DAVID EUGENE SMITH.

Leçons de Géométrie Supérieure. Professées en 1905–1906 par M. E. VESSIOT. Lyon, Delaroche et Schneider, 1906. 4to., 326 pp. (autographed).

THESE lectures delivered by Vessiot during the year 1905–1906 were published in the present form at the demand of his students. The author remarks in the preface that he is hopeful that they may be of service to those who are beginning the study of higher geometry and may serve them as a good preparation for the reading of original memoirs and such works as Darboux's Théorie des surfaces. It is the opinion of the reviewer that the lectures serve these purposes admirably. The attack is direct and the end to be reached is kept clearly before the reader, in fact the whole presentation is such as to lead the beginner to an appreciation of the subject. A glance at the table of contents will convince one that the book will serve as a good introduction to the study of Darboux.

The principal object of the lessons is the study of systems of straight lines but owing to the close relation between lines and spheres it is quite natural that systems of spheres should be studied also. It is assumed at the outset that the student is familiar with the elementary notions of twisted curves and surfaces (tangent planes, tangent lines, etc.), and that he has some acquaintance with the elements of the theory of contact.

In Chapter I, Frenet's formulas for twisted curves are derived and the simple properties of developable surfaces obtained. The rectifying and polar surfaces are discussed as examples of developables. Chapters II, III, and IV are devoted to the general surface theory. Throughout these chapters the importance of the two differential forms of Gauss

$$\Phi(du, dv) = Edu^2 + 2Fdu\,dv + Gdv^2,$$
$$\Psi(du, dv) = E'du^2 + 2F'du\,dv + G'dv^2$$

is insisted upon quite emphatically. These chapters are only introductory to the discussion of systems of lines and spheres, so quite naturally conjugate directions, asymptotic lines, geodesic lines and lines of curvature are the principal subjects treated. In these chapters good geometric interpretations are given to nearly all the analytic results. In Chapter V the foregoing theory is applied to scrolls and developable surfaces and the general properties of these surfaces are derived. Chapters VI, VII, and VIII are devoted to the study of congruences of lines and the correspondence set up by such congruences. Chapter VII is given up entirely to normal congruences and their applications. In this chapter the close relation between lines and spheres is first pointed out and discussed in an elementary manner. The analogy between asymptotic lines and lines of curvature is also pointed out. In Chapter VIII homogeneous and tangential coordinates are introduced. Here the lines of the congruence are defined by corresponding points on two surfaces between which a correspondence has been established. The dual of any line of the congruence is the line of intersection of the tangent planes at the points which define the line of the congruence. Chapters IX and X treat the general and linear line complex; curves and surfaces belonging to a congruence are the principal subjects discussed. Chapter IX is devoted to dualistic transformations and the transformation of Lie. A short discussion of contact transformation is given and then applied to dualistic transformations and the line-sphere transformation of Lie. In this chapter the correspondence between lines and spheres is again taken up and discussed more in detail by means of the Lie transformation.

Chapter XII deals with triply orthogonal systems. After demonstrating Dupin's theorem: On each surface of a triply orthogonal system the intersections with the other surfaces of this system are lines of curvature, — it is applied to the discussion of triply orthogonal systems which contain given surfaces as part of the system. Chapter XIII discusses congruences of spheres and cyclical systems. In this the concluding chapter spheres are discussed without reference to Lie's transformation. At the beginning of the discussion of focal points of a congruence of spheres another theorem of Dupin is proved: A normal congruence of lines is reflected or refracted on any surface into a normal congruence. It is shown that each sphere of the congruence touches the focal surface in two points. The congruence of lines formed by joining these points is discussed and some very pretty relations between the surface of centers of the spheres and their envelope are derived therefrom. The chapter closes with a short account of the cyclical systems of Ribaucour and Weingarten surfaces.

An excellent set of exercises is given to accompany each chapter. C. L. E. MOORE.

Nichteuklidische Geometrie. Von HEINRICH LIEBMANN. Leipzig, G. J. Göschen (Sammlung Schubert, XLIX), 1905. 12mo. viii + 248 pp.

IN this volume of the Schubert collection, Professor Liebmann has succeeded in presenting an introduction to non-euclidean geometry that is brief, readable, and well-balanced. Its brevity will recommend it to the student whose interest in the subject has been aroused by the numerous references in literature, but whose time and maturity are scarcely sufficient for a study of the many longer and more difficult works. It might well appeal also to a teacher of elementary geometry. The recent literature on non-euclidean geometry naturally falls into two classes: the one dealing with the lives and writings of Lobachevsky, Bolyai, and Gauss; and the other consisting of systematic developments of particular phases of the subject. In this book there is a happy combination of the two methods, giving a broad outlook, and yet not sacrificing the unity.

The first chapter contains an interesting account of the parallel axiom and of the attempts at its proof, considered from an historical point of view. The next five chapters, comprising three fourths of the book by pages, are devoted to hyperbolic geometry, beginning with a very simple account of its picturing by means of circles in the euclidean plane. One regrets that references are not given here to some, at least, of the articles that have appeared during the last twenty years on this picturing. With this one exception, the many exact references to a comparatively wide range of literature form one of the most attractive features of the book. There are other chapters on hyperbolic geometry, dealing with the synthetic and the analytic geometry and the trigonometry in the hyperbolic plane. After this more detailed account, the spherical-elliptical geometry is disposed of in one short chapter. The last chapter is on non-euclidean mechanics; and the book is concluded by a few pages in which attention is directed to some of the discussions about the nature of actual space.

E. B. COWLEY.

A Brief Introduction to the Infinitesimal Calculus. Designed especially to aid in reading mathematical economics and statistics. By IRVING FISHER, Ph.D., Professor of political economy in Yale University. Second edition. New York and London, The Macmillan Company, 1906. 12mo. xiii + 84 pp. Price, 75 cents.

THIS book gives an excellent bird's-eye view of the differential calculus, and indeed of the integral calculus. It is written with remarkable clearness, the illustrations from geometry, physics and economics being well chosen and well placed. In this, the second edition, the notion of the "little zero" is not used. Its use in the first edition was criticized by Professor Fiske in his review of the book in the BULLETIN, February, 1898, page 238.

Though small, the book is very comprehensive. If it were to be enlarged, the first addition would perhaps be an article on the mean value theorem, of which article 69 is suggestive, and a page or two on integration as summation, in place of the two short articles 76, 87. Some footnotes, such as the one inserted in the German edition (Teubner, 1904) for article 35, would add to the logical completeness of the proofs, and a few slight changes might be made in the introductory chapter.

The book contains about 200 well selected problems, and is an admirable text-book. It supplies the need, felt by some, of a text-book for those who wish to become familiar, in a short time, with the fundamental conceptions of the calculus.

EDWARD L. DODD.

Leçons sur les Fonctions Discontinues. Par RENÉ BAIRE. Rédigées par A. DENJOY. Paris, Gauthier-Villars, 1905. 8vo. viii + 127 pp.

IN these Leçons the Borel series of monographs has given us a work of fundamental importance in a too long neglected field. The interest in discontinuous functions is happily increasing, and finds in this little book a basis for attack and for classification, so essential if we are not to grope blindly. The following theorem gives the main results:*

A necessary and sufficient condition that a function be the limit of continuous functions is that it be pointwise discontinuous on every perfect assemblage.

Attention should also be called to the concept of semicontinuity, introduced in the proof of this theorem. The limit of the maximum values taken on by a limited function f in intervals (C, D) containing the point A, as (C, D) approaches zero, is called the *maximum* of f at A. If this equals the functional value at the point A, f is said to possess *upper semi-continuity* at A.

The book closes with a general classification of functions. Continuous functions constitute class 0; limits of continuous functions, not themselves continuous, form class 1; in general, limits of functions of class n, not belonging to class n, form class $n + 1$. No attempt is made to show that there exist functions in each class.†

In accordance with the general plan of the series, no advanced mathematical knowledge on the part of the reader is presupposed; this has fortunately compelled the author to give us a very clear and elegant treatment of certain portions of the theories of assemblages and transfinite numbers.

W. D. A. WESTFALL.

Differential Equations. By D. F. CAMPBELL, Armour Institute of Technology. New York, The Macmillan Company, 1906. 96 pp.

THIS is a text-book intended to give a student of engineering a short, practical course in solving differential equations. The author gives only the most common, straightforward methods and omits integrating factors, singular solutions, geometric interpretations, etc.

The first chapter recalls some theorems of algebra and calculus and gives the derivation of a differential equation from its primitive. The use of j instead of i to represent $\sqrt{-1}$ can hardly be considered an improvement. The second chapter deals with changes of the variable (which are practically never used in the rest of the book). It might well have given place to a short explanation of the geometric interpretation of differential equations. In chapter 3, equations of the first order and first degree are discussed. Only the more important types are treated — those which are linear, or can be made so; those which are homogeneous in x and y, or can be made so; those in which the variables are separable; and those which are exact.

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* The proof has been simplified by H. Lebesgue, Leçons sur les fonctions de variables réelles par É. Borel, note 2.

† Lebesgue has proved that functions exist in each class, *Journal de Mathématiques*, 1905.

In the chapter on linear equations with constant coefficients, the usual practice is followed of taking a value of y and proving that it satisfies the equation. In the case of a repeated root of the auxiliary equation, why not follow the same plan? Put $y = xe^{m_1x}$ and show that the equation is satisfied if m_1 is a multiple root. When the $D - a$ operator is introduced, the author goes more slowly than most writers and spends more time than usual in proving the fundamental properties. This should be an improvement, from a pedagogic standpoint. A number of practical examples are added at the end of this chapter. A solitary use for chapter 2 is found in the type of equation so unfortunately called "homogeneous linear." These, together with equations lacking x or y explicitly, are all the equations of order higher than the first which are dealt with. A short chapter is inserted dealing with integrable equations containing more than two variables, and the book closes with a chapter on partial differential equations. No attention is given to systems of equations save that two sample problems are worked; only six examples are given to be solved, so that the last chapter hardly justifies its presence.

Throughout the book the exercises are collected at the end of the chapters, but are arranged in classes, thus giving the student little practice in detecting the type to which any particular equation belongs. More might have been added in a number of places.

A number of misprints have been allowed to creep in. In § 12, c^a should be ce^a; in § 57, $z = 0$ should be $u = c'$; at the end of § 65, "linear" should be omitted; the "familiar theorem of algebra" in example 2, § 69, needs amending. On page 33, sin $x y$, cos $x y$, etc., are used when it would be better to put y sin x, y cos x, etc.

C. R. MACINNES.

Differential Equations. By A. COHEN, Johns Hopkins University. New York, D. C. Heath and Company, 1906. 270 pp.

WHAT from the outside looks like a small text book on differential equations turns out to be a much fuller discussion of the subject. By using good thin paper and eliminating unnecessary margin space, the author has given, in convenient size, quite as full a treatment as either Johnson or Murray. The mechanical work has been well done; the book is well printed, well bound and the proofreading has been carefully done. Even the answers to the problems are accurate as far as they have been tested. A very useful feature of the book is a summary at the end of each chapter of the ground covered in the chapter.

The book opens in the usual way with a chapter on the formation of differential equations. Then equations of the first order and their geometric interpretation are taken up. Before going into equations of order higher than the first, the author introduces (with a question mark) a chapter on total differential equations with three or more variables. Many will feel like emphasizing the question mark. The author's arrangement has the advantage of giving the student miscellaneous exercises of the first order and first degree. It should tend to systematize his knowledge of this fundamental class.

In the chapter on linear equations with constant coefficients, there is a pretty general discussion of the methods of finding the particular integral. In addition to the two ways of breaking up $1/f(D)$, there are also the methods of variation of the parameters and of undetermined coefficients. No mention is made of the short methods of evaluating $e^{ax}/f(D)$, $\sin ax/f(D^2)$, etc., these being replaced by the method of undetermined coefficients. In this Dr. Cohen has taken a step forward. As treated by him, any linear equation with constant coefficients can be solved without the use of integration, provided the right hand member is made up of terms having a finite number of distinct derivatives. It is hardly as immediate as one or two of the short methods, but the advantage of having only a single method to carry in mind counterbalances this. It also puts into one general class all the equations to which the method is applicable.

Very little is given on second order equations, most of the special methods for these being in the chapter on equations of any order. A good chapter on integration in series is brought in, though it might have been moved forward to let the chapter on systems of equations come closer to partial differential equations.

The book would have been improved, I think, by the addition of some easier examples. Those given work out very nicely but require some skill and accuracy in algebra. No misprints of any consequence were found save on page 15, Example 6, *ay* is put instead of *dy*. Many interesting little historical notes have been inserted and references have also been added freely. C. R. MACINNES.

Elements of the Kinematics of a Point and the Rational Mechanics of a Particle. By G. O. JAMES. John Wiley and Sons, New York, 1905. xii + 171 pp.

ON page 102 of this excellent little book the author states its object in the following words: "It is my purpose here to develop in a rigorous manner the elementary theory of the motion of a *material point* or *particle*, and to thereby furnish a point of departure for the consideration of the motion of bodies as they actually occur in the material universe. This elementary exposition will then serve as an introduction to that portion of the subject which is known as *rational mechanics*, and in which the mathematical theory of the motion of portions of matter of *ideal* forms is investigated under *ideal* conditions, leaving the special applications to the particular sciences." The first eight chapters (pages 1–100) are devoted to the kinematics of a point and the last chapters IX to XVII to the mechanics of a free particle. It is not necessary to give in detail the contents of each chapter, some of which are extremely short; it will suffice to state the general structural lines along which the author has laid out the contents of the book. These lines are kept adroitly before the eyes of the reader, who is led straight to the goal and does not lose himself in the detail of side issues. The geometric derivative of a vector and its projection on an arbitrary axis is the fundamental concept upon which velocity and acceleration and their components depend. They are studied in detail both in rectilinear and curvilinear motion, where in each case proper distinction is made between absolute and relative motion; this is next extended to angular and axial motion, where again the same notions come into use which have been developed under linear motion. Here too the periodic form of motion is gone into in some detail. In the last two chapters of this preliminary kinematic study the motion is referred to coordinate axes in space, which are either fixed or movable. Before the second part of the book is discussed a few remarks concerning the first are offered: In the definition of the geometric derivative of a vector the figure 10 is somewhat misleading to a beginner inasmuch as the tangent PP' is here perpendicular to the vector AP which is not necessary. Since the parallelogram of vectors is the foundation of the entire structure it would have helped the understanding to proceed with the derivative of the vector with moving origin, as Budde and others have done. This secures to the beginner a geometric insight and guide which in the study of mechanics is all important. For this reason it would have been desirable to go into the velocities and accelerations for polar coordinates from the standpoint of movable axes and to make plain to the reader the significance of the expression $d^2r/dt^2 - r(d\theta/dt)^2$. The corollary of article 12 which plays such an important part in the book would have gained in lucidity and importance by such a geometric study. In this way the fact that the acceleration vector lies in the osculating plane would have come out with full force and not as a supplementary algebraic result (see §64). For the same geometric reason the notion of areal velocity and areal acceleration might have been replaced by their usual equivalents, especially since the dimensions of these terms differ from those of linear velocity and linear acceleration; in the same way angular velocity and acceleration are concepts which in the mechanics of a particle should have but transitory importance.

The characteristic feature of the second part of the book is found in the fact that the potential (or as the author puts it the field of force) is considered as the primary mechanical concept. Force as such is merely used as an abbreviation in the text. Newton's three fundamental principles are expressed in a fashion consistent with the four fundamental notions of space, time, mass, and field of force. It is to be supposed that the author will publish additional volumes in which the wisdom of employing the field of force as a primary concept in a rather elementary text is made more cogent than would appear from the limited material offered in this book. In conclusion it must be stated again that although the reader might differ from the author in many particulars as to choice of subject matter and form of presentation, he cannot help enjoying the clearness, consistency, and cogency with which the author presents his case from cover to cover.

KURT LAVES.

Cours d'Astronomie. Première Partie: Astronomie théorique. By H. ANDOYER. Paris, Librairie Scientifique A. Hermann, 1906. 222 pp.

AFTER a rather prolonged lull in the publication of textbooks on spherical and practical astronomy we are now about to to receive from the press two treatises on the same subject, the one by Professor W. Foerster, the former eminent director of the observatory of Berlin, the other by Professor Andoyer, the well-known scholar of the Paris observatory. The first parts of both treatises have just left the press. A review of Professor Foerster's book has been published in the *Astrophysical Journal.* In giving a short outline of Professer Andoyer's book in a journal devoted to mathematical science, an effort is made to bring out those points primarly which are of interest to mathematicians. Chauvenet's two large volumes on spherical astronomy are too voluminous to lend themselves easily to the needs of a mathematician who tries to inform himself about the application made in astronomy of a certain mathematical theorem he is interested in. Andoyer's book is much more adapted for such purposes. After an introductory chapter concerning spherical trigonometry and a short deviation into spheroidal trigonometry to the extent to which this is needed for elementary geodesic questions, the author gives in the next seven chapters a rather condensed account of refraction, parallax and aberration. Before the theory of precession and nutation is taken up, the reader is initiated in Chapter IX into the more elementary notions of celestial mechanics. This is necessary since the apparent position of a planet, after it has been corrected for refraction, parallax, and aberration, will yet have to be freed from the disturbing influences of the neighboring bodies. To quote but one example: in the definition of apparent solar time, the center of the earth is pulled out of its elliptic path by the various members of the solar system. These perturbations of the individual members must be brought into tables so that for a given value of the time the amount of pull due to each individual member can be properly added to the position of the earth in the elliptic path. Now since the apparent solar time is measured in the ecliptic, from the point of vernal equinox, *i. e.*, the point where the equator and the ecliptic of a time intersect each other in springtime, it is obvious that the perturbing influence of surrounding matter on the oblate earth is to affect the positions of the planes of reference: equator and ecliptic. These changes are comprised under the name of precession and nutation. In the ninth chapter a short discussion is found concerning the convergence of series used in astronomy. It is the only place in the book where a knowledge beyond the differential and integral calculus is needed. Chapter X is devoted to precession and nutation and formulas are derived for the equatorial and elliptic coordinates to correct the position of an object for the secular and periodic changes in the position of the planes of reference. It would seem as if by the study of the polar triangles the author might have accomplished his result even more successfully, thus keeping the distinction between planetary and luni-solar precession clearly before the eyes of the reader.

The last chapters of the book deal with the geocentric motions of the sun, moon, planets and their satellites. Here the eccentricities and inclinations are assumed to be zero to simplify the deductions. An elementary exposition of the theory of eclipses of moon and sun and of the occultation of stars by the moon finally ends this first part of Professor Andoyer's treatise.

It is perhaps not out of place to point to the great wealth of problems that the subject matter of spherical astronomy affords. In doing so it is but necessary to state in its most general aspect the problem that this science aims to solve: An observer O is located on the surface of a rotating spheroid (earth) the center of which revolves in a well defined manner about the center of gravity of the solar system. A ray of light emanating at the time t_0 from a movable point P (a planet) passes through the atmosphere of the earth and reaches the retina of the observer at the time t_1. We know the heliocentric positions of P and the center of the earth at any time t and we ask to find the *apparent* geocentric position of P at an arbitrarily chosen epoch T. The solution of this problem can be obtained only by putting a number of restrictions on it; to mention but one, we have to keep the interval of time $T - t$ sufficiently small in order to hold the difference between the observed and computed position below an assigned quantity. The contents of the various chapters of the book in question furnish in turn the steps by which the solution of this problem is accomplished.

KURT LAVES.

Vorlesungen über technische Mechanik. Von AUGUST FÖPPL. Dritte Auflage. Bd. 1: *Einführung in die Mechanik*, xvi + 428 pp.; Bd. 3: *Festigkeitslehre*, xvi + 434 pp. Leipzig, B. G. Teubner, 1905.

THE valuable and highly popular work of Föppl on technical mechanics, which began its publication about ten years ago and rapidly ran into a second edition, is now appearing in a third edition of which the first and third volumes are already printed. The present plan calls for no serious changes in the text* other than an expansion into five volumes, the last of which is to contain a considerable amount of matter important for students of technical mechanics but somewhat more advanced than properly finds a place in the general fundamental lectures which fill the first four volumes.

The first or introductory volume still adheres closely to the original maxims of the author, namely, that mechanics is in reality a branch of physics and should be thus presented to beginners, and that no material should be inserted merely because some persons of special and restricted point of view might call for it. One of the most admirable features of the volume is its presentation of the elements of elasticity and hydromechanics in addition to the discussion of the mechanics of a particle and rigid body. It is probably true, and as such it is certainly regrettable, that the great majority of students leave the subject of mechanics after a course by no means meagre with the conviction that mechanics means merely the equilibrium and motion of a particle or rigid body. There is no great difficulty in giving even beginners a realizing sense of the fact that the subject is broader, that the general laws are equally applicable to the study of continuous distributions of matter. This, however, can only be accomplished by thorough adherence to the principle that mechanics is physics rather than mathematics and by a sacrifice of problems which involve complicated mathematical treatment. This the author does with great discretion. His students need not be experts at calculus and analytic geometry to be able to follow with appreciation the treatment of questions taken up in the introductory volume.

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* An extended review of the four volumes as they appeared in the second edition was given in the BULLETIN, volume 9, pp. 25-35, 1902.

Even in the third volume, which especially deals with elasticity, the analysis is kept in the background wherever possible, although toward the close where the subject of wave motion in elastic media is treated and considerable mathematical machinery becomes necessary there is no shrinking from mathematical complications. Such choice in analysis, setting it aside where it can well be done away and unhesitatingly introducing it where it becomes advisable, is one of the most valuable characteristics of the book — and the more valuable as it is so rare in the majority of books with which we are acquainted. Notwithstanding the introduction of some new material the size of the volume has been diminished by eighty pages. The reduction has been possible partly through the elimination of material that seems less vital, partly through the reservation of some subjects for the contemplated fifth volume. The appearance of this supplementary part of the whole work will be awaited with interest by all who are concerned with the question of the proper presentation of mechanics as a whole.

E. B. WILSON.

Carl Friedrich Gauss Werke. Band VII. Theoria Motus Corporum Cœlestium in Sectionibus Conicis Solem Ambientium. Herausgegeben von der Königlichen Gesellschaft der Wissenschaften zu Göttingen. Leipzig, B. G. Teubner, 1906. Pp. 650.

IT was within two or three years of a century ago that Gauss's famous Theoria motus first appeared. Now it comes out as the first 290 pages of the seventh volume of his complete works. The remaining 360 pages of the volume are made up of various notes and letters, in small type, which have been culled from the huge Nachlass. To rank all this matter as notes would, however, be extreme minimization; for there are two extensive investigations on the perturbations of Ceres and Pallas filling respectively 35 and 200 pages. Of these the latter for a long time seemed destined to receive the large prize offered by the French academy for a treatment of the perturbations of the asteroid Pallas; but like so much of the work of this Princeps mathematicorum, it never came to publication during his life, which lasted some twenty years after the investigation was practically completed. Nothing shows more clearly the great assiduity and rapidity of Gauss's researches than the way he carried through his computations on Pallas to the extent of over 340,000 figures in the time of about three months.

With this volume VII the edition of Gauss's works, which has occupied nearly forty years in publication, is complete except for one miscellaneous volume, the tenth, which will probably soon be given to the public. The scientific world owes a deep debt of gratitude to those who have so cheerfully spent a large amount of their time and energy on this great and highly valuable undertaking.

E. B. WILSON.

NOTES.

THE July number (volume 29, number 3) of the *American Journal of Mathematics* contains the following papers: "Concerning a certain type of continued fractions depending on a variable parameter," by T. E. MCKINNEY; "Twisted curves whose tangents belong to a linear complex," by V. SNYDER; "Groups in which every subgroup is either abelian or dihedral," by G. A. MILLER; "Lines of curvature of a surface," and "The ovals of the plane sextic curve," by J. E. WRIGHT.

AT the meeting of the London mathematical society held on May 9, the following papers were read: By H. F. BAKER, "Rational expression of the invariants of a quintic by means of three"; by H. LAMB, "Secular stability"; by F. J. W. WHIPPLE, "A lemma connected with Fourier's series."

AT the meeting held on June 13 the following papers were read: By A. R. FORSYTH, "Note on a special set of classes of partial differential equations of the second order"; by T. J. I'A. BROMWICH, "Various extensions of Abel's lemma"; by J. W. L. GLAISHER, "On the number of representations of a number as a sum of $2r$ squares, when $2r$ does not exceed 18"; by A. E. WESTERN, "An extension of Eisenstein's law of reciprocity"; by A. B. BASSET," "On certain singular points of surfaces"; by E. B. ELLIOTT, "The minimum necessary postulates as to a function to be defined as analytic over a region."

THE last list of members of the Société mathématique de France, published in January, contains 280 names, of which 20 are Americans. The meetings of the society are held fortnightly except during the summer vacation. The society was founded in 1872, and at once began the publication of its *Bulletin*, the first volume appearing in 1873. A volume has been issued every year since, the parts appearing quarterly. There are 75 journals on its exchange list. Instead of having a special editor, the *Bulletin* is published by the secretaries, who are elected annually. The officers for the present year are: president, Dr. E. BLUTEL; secretaries, Dr. S. SERVANT and Professor L. RAFFY.

THE following mathematical works are announced to be in the press of the Carnegie Institution of Washington and will appear in a few weeks: "Synopsis of linear associative algebra. A report on its natural development and the results reached up to the present time," by J. B. SHAW; "Researches on the performance of the screw propeller," by W. F. DURAND; "Dynamical meteorology and hydrography," by V. BJERKNES and J. W. SANDSTRÖM.

THE following parts of the Encyklopädie der mathematischen Wissenschaften are announced by Teubner of Leipzig to be in the press, and will probably appear during the summer: II, 2, Elliptische Funktionen, by J. HARKNESS and W. WIRTINGER; Automorphe Funktionen, by R. FRICKE; III, 1, Beziehung und Gegensatz von synthetischer und analytischer Geometrie in seiner historischen Entwicklung im neunzehnten Jahrhundert, by G. FANO; Die Gruppentheorie als geometrisches Einteilungsprinzip, by G. FANO; IV, 1, Die elementare Dynamik, by P. STÄCKEL, with additions by J. PETERSEN; IV, 2, Theorie des Schiffes, by A. KRILOFF; Grundlegung der mathematischen Elastizitätslehre, by C. H. MÜLLER and A. TIMPE; Specielle Ausführungen zur Statik elastischer Körper, by O. TEDONE and A. TIMPE; Schwingungen elastischer Körper, insbesondere Akustik, by H. LAMB; VI, 1, Höhere Geodäsie, by P. PIZZETTI.

Of the French edition of the encyclopædia, $\mathrm{I}_{1,\ (2)}$ containing Nombres irrationnels et notion de limite, and algorithmes illimités by A. PRINGSHEIM and J. MOLK have just appeared. The details of the editorship of IV, mechanics, have been made public; it is to be published in five volumes under the general direction of Professors Appell and Molk.

THE mathematical and physical section of the Royal society of Naples announces the following prize problem: "A systematic exposition of the results thus far obtained on geometric configurations of the plane and of higher spaces, putting them in relation with the theory of substitutions, and containing if possible some new contribution." Competing memoirs must be written in Italian, Latin or French, and submitted to the secretary not later than June 30, 1908. The prize of 500 lire will be awarded on the first Sunday of 1909.

THE Royal academy of Belgium announces the following prize problems for 1908:

1. Make a systematic exposition of researches already made of critical phenomena in physics. Complete our knowledge of this question by new researches. Prize of 800 francs.
2. An important contribution to the study of the differential equation $Xdx + Ydy = 0$ is desired, X, Y being known quadratic functions of x, y. Prize of 800 francs.
3. New researches on the calorific conductibility of liquids and solutions is desired. Prize of 800 francs.
4. Make a historical and critical study of the experiments on unipolar induction of Weber, and elucidate by means of new experiments, the laws and the interpretation of the physical fact. Prize of 800 francs.
5. Systematize and complete the investigations made in the calculus of variations since 1850. Prize of 600 francs.

Memoirs should be written in French or Flemish and sent to the secretary before August 1, 1908.

THE prize problem of the Prince Jablonowski society of Leipzig for 1910 is as follows: "Most problems in electrostatics are reducible to the determination of the Green distributions of masses, and consequently these distributions are of primary importance for the theory of electrostatics and for the entire potential theory. From recent memoirs (*Leipziger Berichte* for 1906, pages 483–558) it can doubtless be concluded that in the theory of the logarithmic potential for any closed curve the two Green distributions corresponding to the inner and outer spaces can be reduced to a single distribution, the so-called 'fundamental distribution,' and that an analogous property exists in the theory of the newtonian potential for any closed surface. However, in the memoirs mentioned, much is still left to be desired; accordingly the society proposes the following problem: A memoir is desired, in which the theory of the 'fundamental distribution,' either in clearness and rigor or in extent and completeness, shall be considerably advanced." The prize is 1,500 marks.

THE following advanced courses in mathematics are announced for the academic year 1907–1908:

UNIVERSITY OF CHICAGO (October 1 to June 15). The total number of hours is indicated. — By Professor E. H. MOORE: Selected chapters in analysis, 48 hours; Theory of functions of a complex variable, 24 hours; Seminar, theory of functions of a real variable, 24 hours. — By Professor O. BOLZA: Advanced integral calculus, 96 hours; Calculus of variations, 96 hours. — By Professor H. MASCHKE: Solid analytics and determinants, 48 hours; Algebraic analysis, 48 hours; Projective synthetic geometry, 48 hours; Differential geometry, 96 hours; Partial differential equations, 48 hours. — By Professor L. E. DICKSON: Finite groups with applications to algebra and linear substitution groups, 96 hours. — By Professor H. E. SLAUGHT: Differential equations with applications, 48 hours. — By Professor J. W. A. YOUNG: Introduction to the theory of numbers, 48 hours. — By Dr. A. C. LUNN: Analytic mechanics, 48 hours. — By Professor K. LAVES: Analytic mechanics, 96 hours. — By Professor F. R. MOULTON: Introduction to celestial mechanics, 96 hours; Planetary perturbations, 96 hours.

HARVARD UNIVERSITY. — By Professor W. E. BYERLY: Differential and integral calculus (second course), three hours; Trigonometric series, three hours (with Professor Peirce). — By Professor B. O. PEIRCE: Hydromechanics, three hours. — By Professor W. F. OSGOOD: Elements of mechanics, three hours; Infinite series and products (first half year), three hours; Theory of functions of a complex variable (second half year), three hours. — By Professor M. BÔCHER: Introduction to modern geometry and modern algebra, three hours; Vector analysis and quaternions, three hours; The properties of polynomials (first half year), three hours; Definite integrals and integral equations (second half year), three hours. — by Professor C. L. BOUTON: Elementary theory of differential equations (second half year), three hours: Geometric transformations, three hours. — By Professor J. K. WHITTEMORE: Theory of functions I, three hours; Theory of the figure of the earth (second half year), three hours. — By Dr. J. L. COOLIDGE: Algebraic plane curves, three hours.

UNIVERSITY OF ILLINOIS.— By Professor S. W. SHATTUCK: Differential equations and calculus of variations, three hours. — By Professor E. J. TOWNSEND: Theory of functions, three hours; Seminar, two hours. — By Professor G. A. MILLER: Theory of numbers, three hours; Theory of determinants, two hours. — By Professor H. L. RIETZ: Theory of averages and actuarial theory, three hours. — By Professor J. STEBBINS: Method of least squares, two hours. — By Professor C. N. HASKINS: Solid analytic geometry, three hours; Spherical harmonics and the potential function, three hours. — By Miss M. B. WHITE: Teacher's course, two hours. — By Dr. L. I. NEIKIRK: Theory of equations, three hours. — By Dr. C. H. SISAM: Modern geometry and algebraic surfaces, three hours. — By Dr. A. R. CRATHORNE: Partial differential equations, two hours.

INDIANA UNIVERSITY. — By Professor R. J. ALEY: Theory of numbers, two hours; Differential equations, three hours (autumn, winter); Mathematical pedagogy, two hours (s) — By Professor S. C. DAVISSON: Modern analytic geometry, two hours (a, w); Theory of surfaces, two hours; Non-euclidean geometry, two hours (w, s). — By Professor D. A. ROTHROCK: Advanced calculus, three hours; Quaternions with applications, three hours (a, w); Potential functions, two hours (w, s). — By Professor U. S. HANNA: Substitution groups, three hours (a); Galois theory of equations, three hours (w). — By Dr. C. HASEMAN: Partial differential equations, three hours.

OXFORD UNIVERSITY (*Michaelmas Term*). — By Professor W. ESSON: Analytic geometry of plane curves, two hours; Synthetic geometry of plane curves, one hour. — By Professor E. B. ELLIOTT: Sequences and series, two hours; Elementary theory of numbers, one hour. — By Professor A. E. H. LOVE: Magnetism and electricity, three hours. — By Professor H. H. TURNER: Elementary mathematical astronomy, two hours. — By Mr. A. L. PEDDER: Problems in pure mathematics, one hour. — By Mr. J. E. CAMPBELL: Differential equations, two hours. — By Mr. C. H. THOMPSON: Integral calculus, two hours. — By Mr. E. H. HAYES: Analytic statics, two hours. By Mr. A. L. DIXON: Hydrostatics, one hour. — By Mr. H. T. GERRANS: Tridimensional rigid dynamics, two hours. — By Mr. C. E. HASELFOOT: Theory of equations, one hour. — By Mr. P. J. KIRKBY: Projective geometry, two hours. — By

Mr. A. E. JOLLIFFE: Analytic geometry, two hours. — By Mr. J. W. RUSSELL: Differential calculus, two hours. — By Mr. R. F. MCNEILE: Curve tracing, one hour. — By Mr. C. H. SAMPSON: Solid geometry, one hour.

UNIVERSITY OF BOLOGNA. — By Professor C. ARZELÀ: Lebesgue's integrals, Dirichlet's principle, Fourier's series, three hours. — By Professor L. DONATI: Electromagnetic fields and dynamics of electrons, three hours. — By Professor L. PINCHERLE: Analytic functions; algebraic functions and their integrals, three hours.

UNIVERSITY OF CATANIA. — By Professor G. LAURICELLA: Theory of heat and propagation of waves, four and a half hours. — By Professor G. PENNACCHIETTI: Advanced kinematics and rigid dynamics, four and a half hours. — By Professor M. PIERI: Projective geometry of hyperspace, three hours. — By Professor C. SEVERINI: Finite continuous groups of transformations, contact transformations, four and a half hours.

UNIVERSITY OF GENOA. — By Professor G. FUBINI: Existence theorems of the calculus of variations, three hours. — By Professor G. LORIA: Plane representation of algebraic surfaces, rational transformations of the plane and space, three hours. — By Professor O. TEDONE: Spherical functions, Lamé's functions with applications to electro- and magnetic statics, three hours.

UNIVERSITY OF MESSINA. — By Professor G. BAGNERA: Theta functions of several arguments and relative groups of characteristics, three hours. — By Professor R. MARCOLONGO: Electric and optical phenomena in moving media, three hours. — By Professor V. MARTINETTI: Projective geometry of hyperspace, three hours — By Professor L. ORLANDO: Definite integrals with applications to mathematical physics, three hours; Principles of the theory of numbers, two hours. — By Professor G. VIVANTI: Calculus of variations, three hours.

UNIVERSITY OF NAPLES. — By Professor F. AMODEO: History of mathematics during the last three centuries, three hours. — By Professor A. CAPELLI: Theory of groups with analytic applications, four and a half hours. — By Professor P. DELL PEZZO: Analytic functions, Riemann surfaces and automorphic functions, four and a half hours. — By Professor D. MONTESANO: Theory of hyperspaces and birational transformation, four and a half hours. — By Professor L. PINTO: Physical optics, four and a half hours.

UNIVERSITY OF PADUA. — By Professor F. D'ARCAIS: Differential equations and theory of functions of a complex variable, four and a half hours. — By Professor A. FAVARO: History of Italian mathematics in the sixteenth and seventeenth centuries, three hours. — By Professor P. GAZZANIGA: Theory of numbers, three hours. — By Professor T. LEVI-CIVITA: Differential equations of dynamics; contact transformations with applications to dynamics and optics, four and a half hours. — By Professor G. RICCI: Theory of potential, electro- and magneto-statics, four hours. — By Professer F. SEVERI: Advanced theory of algebraic functions of one and two variables, three hours. — By Professor G. VERONESE: Geometry of hyperspace, four hours.

UNIVERSITY OF PALERMO. — By Professor F. GERBALDI: Differential geometry, four and a half hours. — By Professor G. B. GUCCIA: General theory of algebraic curves and surfaces, four and a half hours. — By Professor G. TORELLI: Mathematical theory of elasticity, four and a half hours. — By Professor A. VENTURI: Rotation of solid bodies, application to the earth, precession, nutation and motion of the pole, four and a half. hours.

UNIVERSITY OF PAVIA. — By Professor E. ALMANSI: Laplace's equation with applications to theoretical physics, three hours. — By Professor L. BERZOLARI: Algebraic forms with geometric applications, three hours. — By Professor E. PASCAL: Contact transformations with applications, three hours.

UNIVERSITY OF PISA. — By Professor E. BERTINI: Hyperspaces, geometry on an algebraic curve with applications, four and a half hours. — By Professor L. BIANCHI: Differential geometry of curves and surfaces, surfaces applicable to quadrics, four and a half hours. — By Professor U. DINI: Harmonic functions and Fourier's series, four hours. — By Professor G. A. MAGGI: Equilibrium and motion of elastic solid bodies, with applications to optics, four and a half hours. — By Professor P. PIZZETTI: General theory of planetary perturbations, three hours.

UNIVERSITY OF ROME. — By Professor G. BISCONCINI: Mathematical theory of elasticity, with technical applications, three hours. — By Professor G. CASTELNUOVO: Geometry on an algebraic surface, three hours. — By Professor V. CERRUTI: Calculus of variations, with applications to geometry and mechanics, three hours. — By Professor V. VOLTERRA: Electromagnetic fields, four and a half hours: The problem of three bodies, three hours.

UNIVERSITY OF TURIN. — By Professor T. BOGGIO: Applications of Fredholm's integral equations to mathematical physics, three hours. — By Professor E. D'OVIDIO: Theory of algebraic forms, three hours. — By Professor G. MORERA: Newtonian forces, equilibrium of a rotating liquid, three hours. — By Professor C. SEGRE: Selected chapters of line geometry, three hours. — By Professor C. SOMIGLIANA: Propagation of heat, kinetic theory of gases, three hours.

WITH the aid of a gift from Mr. C. S. BARTON, Columbia University has secured a very extensive collection of mathematical models and instruments for a mathematical laboratory and museum.

IN Erlangen, on the twenty-seventh of April, the seventieth birthday of Professor PAUL GORDAN was appropriately celebrated by an elaborate programme, including numerous addresses from associates and collaborators. Professor Gordan will retire from active service at the close of the present semester.

PROFESSOR E. SELLING, of the University of Würzburg, has retired from active teaching.

MR. ARTHUR HOLDEN has been appointed assistant lecturer and tutor in mathematics at the University of Sheffield.

THE Accademia dei Lincei of Rome, has divided the royal prize of 10,000 lire for mathematics equally between Professor C. ARZELÀ, of the University of Bologna, and Professor G. CASTELNUOVO, of the University of Rome.

THE Italian scientific society has awarded its gold medal for researches in mathematics to Professor G. LAURICELLA, of the University of Catania.

MR. H. BATEMAN, lecturer in the University of Liverpool, has been appointed reader in mathematical physics at the University of Manchester. MR. J. E. LITTLEWOOD has been appointed lecturer in mathematics and MR. H. M. PRIESTLEY assistant lecturer in mathematics in the latter university.

At the University of Michigan, Professor J. L. Markley has been promoted to a full professorship of mathematics. Dr. P. Field, Dr. W. B. Ford and Dr. T. R. Running have been promoted to assistant professorships of mathematics. Dr. J. W. Glover, professor of mathematics and insurance, has spent the past year, on leave of absence, as consulting actuary of the Wisconsin legislative committee on banks and insurance.

Dr. C. C. Grove, of Hamilton College, has been appointed professor of mathematics in Roanoke College.

At the Massachusetts Institute of Technology, Professor F. H. Bailey has been promoted to a full professorship of mathematics, and Mr. L. M. Passano to an assistant professorship.

Professor H. F. Stecker, of Pennsylvania State College, has been promoted to an associate professorship of mathematics.

At Cornell University Mr. C. F. Craig and Dr. F. W. Owens have been appointed instructors in mathematics.

At the University of Illinois, Dr. A. R. Crathorne has been appointed instructor in mathematics, and Mr. H. P. Kean, and Mr. W. W. Denton assistants in mathematics. Dr. E. L. Dodd has been appointed instructor in mathematics at the University of Texas.

Dr. A. L. Underhill, of Princeton University, has been appointed instructor in mathematics at the University of Wisconsin.

Mr. Joseph Lipke, of Columbia University, has been appointed instructor in mathematics at the University of California.

At Vassar College, Miss E. Morenus has been appointed instructor in mathematics; Miss G. Smith has been granted a leave of absence and will spend next year in study at Paris.

Dr. Edward John Routh, the well-known mathematician of Cambridge, died June 7, 1907, at the age of 76 years. He was author of several works on dynamics and formerly lecturer of mathematics at Peterhouse and Pembroke Colleges.

Professor Francesco Siacci, senator of the kingdom of Italy and professor of theoretical mechanics at the University of Naples, died May 31, at the age of 68 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Albrich (K.). Behandlung der Funktionen im Mittelschulunterricht. (Progr.) Hermannstadt, 1906. 8vo. 47 pp.

Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Herausgegeben im Auftrage der Akademien der Wissenschaften zu Gottingen, Leipzig, München und Wien, sowie unter Mitwirkung zahlreicher Fachgenossen. Band V: Physik. Redigiert von A. Sommerfeld. Teil 2, Heft 2. Leipzig, Teubner, 1907. 8vo. Pp. 289–392. M. 3.00

Galdeano (Z. G. de). Tratado de analisis matematica. Tomo V: Applicaciones del calculo infinitesimal al estudio de las figuras en el espacio. Zaragoza, 1906. 4to. 576 pp.

Garbieri (G.). Geometria analitica. Riassunto di lezioni date nell' Università di Genova. Parte I: Luoghi di primo grado, con cenno su altri luoghi in coordinate Cartesiane. Torino, 1907. 8vo. 60 pp.

Klein (H.). Die Dreiersysteme von 13 Elementen. (Progr.) Hermannstadt, 1906. 8vo. 23 pp.

Kosmik (K.). Pythagoreische Dreiecke und diesbezügliche Vierecke. (Progr.) Mahr.-Weisskirchen, 1906. 8vo. 41 pp.

Krug. Die niedere Analysis auf der Unterrichtsstufe des Realgymnasiums. Teil 2. (Progr.) Stuttgart, 1906. 4to. 69 pp.

Lorentz (H. A.). Lehrbuch der Differential- und Integralrechnung, nebst Einfuhrung in andere Teile der Mathematik mit besonderer Berücksichtigung der Bedürfnisse der Studirenden der Naturwissenschaften. Uebersetzt von C. G. Schmidt. 2te Auflage. Leipzig, 1907. 8vo. 6 + 562 pp. M. 12.00

Lubelza (J.). Lecciones de cálculo infinitesimal. Libros II y III: Cálculo integral y aplicaciones. Madrid, 1906. 4to. Pp. 1–22 and 131–382.

Mellor (J. W.). Higher mathematics for students of chemistry and physics. New York, Longmans, 1906. 8vo. Cloth. $4.50

Möbius (P. J.). Ueber die Anlage zur Mathematik. 2te vermehrte und veränderte Auflage. Leipzig, 1907. 8vo. 16 + 264 pp. M. 4.50

Hill (G. W.). The collected mathematical works. Vol. 4. Washington, D. C., Carnegie Institution, 1907. 4to. 6 + 460 pp. Cloth. $2.50

Huntington (E. V.). La kontinuo. Elementa teorio starigita sur la ideo de ordo kun aldono pri transfinitaj nombroj. Tradukita de la anglo lingvo kun la permeso de la aŭtoro de Raoul Bricard. Paris, Gauthier-Villars, 1907. 16mo. 10 + 126 pp. F. 2.75

Papelier (G.). Précis de géométrie analytique à l'usage des élèves de mathématiques spéciales. 2e fascicule. Paris, Vuibert, 1907. 8vo. Pp. 449 à 696.

Roth (C.). Einiges uber ältere und neuere Methoden der Rektifikation sowie der Teilung von Kreisbögen. (Progr.) Mühlbach, 1906. 4to. 22 pp.

Salmon (G.). Analytische Geometrie der Kegelschnitte mit besonderer Berücksichtigung der neueren Methoden. Nach Salmon frei bearbeitet von Wilhelm Fiedler. 7te Auflage. Teil I. Leipzig, Teubner, 1907. 8vo. 35 + 444 pp. Cloth. M. 10.00

Sievert (H.). Die Parameterdarstellung der Kurven 3ter Ordnung durch Thetafunktionen. (Progr.) Bayreuth, 1906. 8vo. 43 pp.

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II. ELEMENTARY MATHEMATICS.

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